LUND UNIVERSITY

2021.03.30

VTAF 01 – Sound in Buildings and Environment

3. Waves in solids

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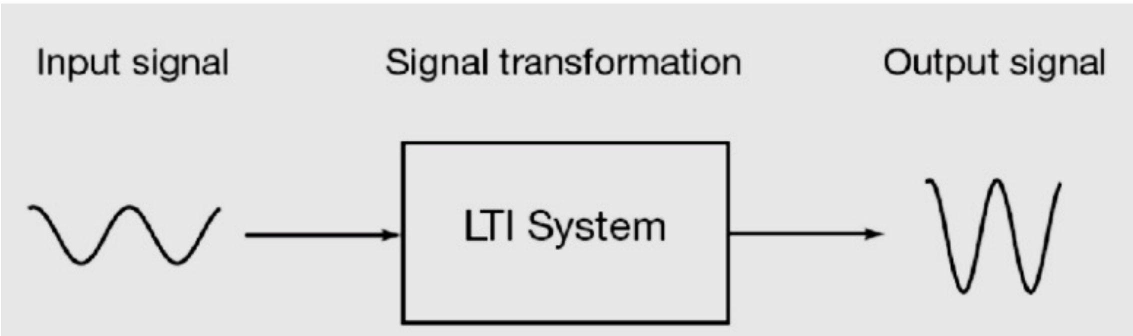
Recap from previous lecture 2

- Equations of motion of
  - Single-degree-of-freedom systems (SDOF)
    - » Damped
    - » Undamped
  - Multi-degrees-of-freedom systems (MDOF)
- Concepts of
  - Eigenfrequency / Eigenmodes
  - Resonances
  - Frequency response functions (FRFs)

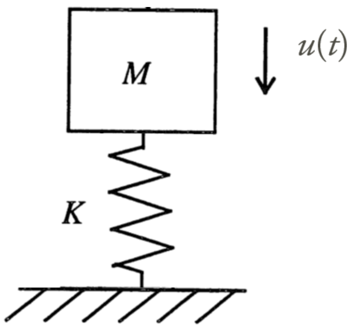
Reminder: Linear systems

- What effect does an input signal have on an output signal?
  - What effect does a force on a body have on its velocity?

- A way to answer it using theory of **linear time-invariant systems**

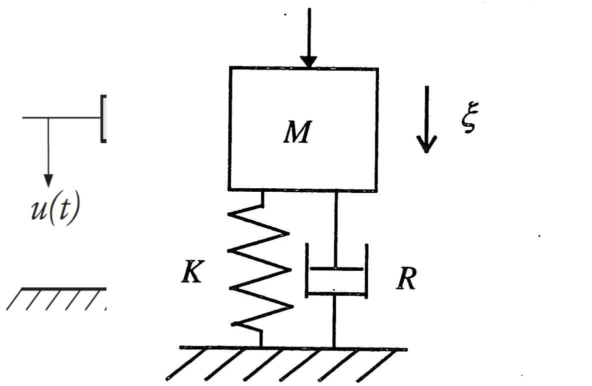


Equations of Motions of a mass-spring system (SDOF)



- Forces in the system:
  - Newton's law: $M\ddot{x}$
  - Hooke's law: $-Kx$
- Forces shall balance each other:
  - $M\ddot{x} = -Kx$
  - $M\ddot{x} + Kx = 0$
- We got equations of motions (EOM) of the system!
- $x(t) = ae^{\lambda t}$
- $\lambda_1 = \mathrm{i}\sqrt{\frac{K}{M}} = \mathrm{i}\omega_0$; $\lambda_2 = -\mathrm{i}\sqrt{\frac{K}{M}} = -\mathrm{i}\omega_0$.
- $x(t) = ae^{\mathrm{i}\omega_0 t} + be^{-\mathrm{i}\omega_0 t} = \mathrm{A}\sin(\omega_0 t) + \mathrm{B}\cos(\omega_0 t)$.

Free vibrations with damping

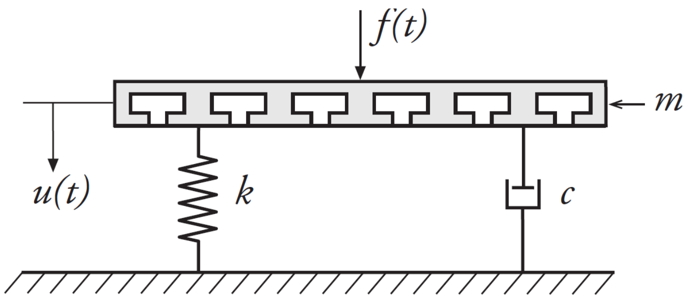


- $\lambda_1 = -\mathrm{R} + \mathrm{i}\omega_R = \mathrm{i}\omega_0\ ;\ \lambda_2 = -\mathrm{R} - \mathrm{i}\omega_R\ ;\ \omega_R = \sqrt{\omega_0^2 - R^2}.$
- $x(t) = ae^{-\mathrm{R}t+\mathrm{i}\omega_R t} + be^{-\mathrm{R}t-\mathrm{i}\omega_R t} = [\mathrm{A}\sin(\omega_R t) + \mathrm{B}\cos(\omega_R t)]\, e^{-\mathrm{R}t}$

Damped SDOF – EOM

- Mass-spring-damper system (e.g. a floor)

$F(t) = F_{driv} \cdot \cos(\omega t)$



$$M\ddot{u}(t) + R\dot{u}(t) + Ku(t) = F(t)$$

Inertial force — Damping force — Elastic force — Applied force

  - $u(t)$ obtained by solving the PDE together with the initial conditions
    - Solution = Homogeneous + Particular

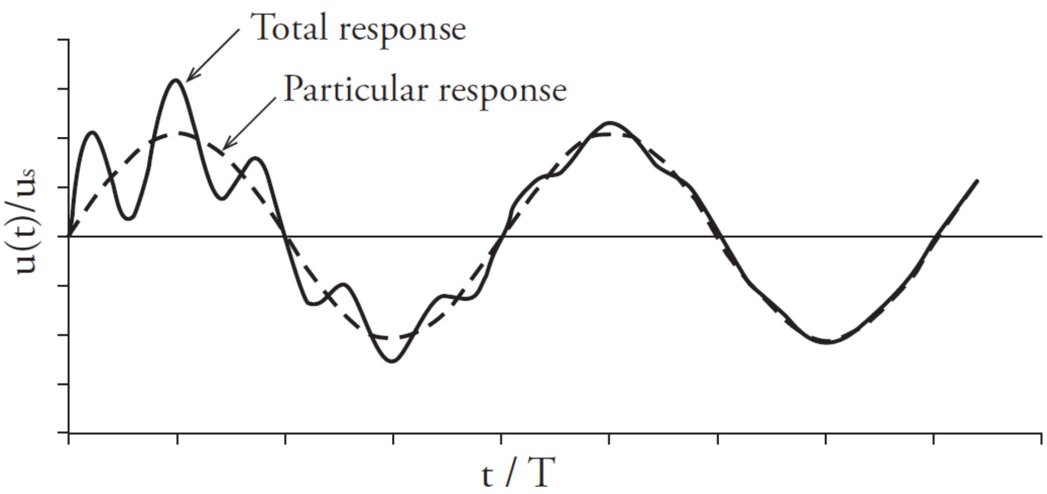
$$u(t) = u_p(t) + u_h(t)$$

$u_p(t)$: $F(t) = F_{driv} \cdot \cos(\omega t)$ — $u_h(t)$: $F(t) = 0$

NOTE: Damping is the energy dissipation of a vibrating system

Damped SDOF – Total solution

- Total solution = homogeneous + particular
  - The homogeneous solution vanishes with increasing time. After some time: $u(t) \approx u_p(t)$



Total response of a damped system subjected to a harmonic force,

$$u_{total}(t) = \underbrace{e^{-\frac{\eta}{2}\omega_0 t}\left(B_1 \sin(\omega_d t) + B_2 \cos(\omega_d t)\right)}_{\text{Homogeneous}} + \underbrace{D_1 \sin(\omega t) + D_2 \cos(\omega t)}_{\text{Particular}}$$

Homogeneous

Particular

$$\omega_d = \omega_0 \sqrt{1 - \xi^2}$$

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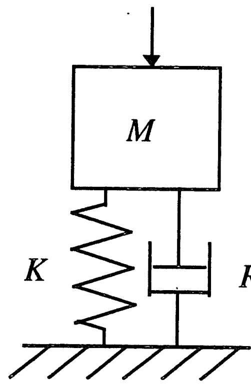
SDOF – Complex representation (Freq. domain)



- Euler's formula: $e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$
- Then: $F(t)=F_{driv}\cos(\omega t) = \mathrm{Re}\left[F_{driv}e^{i\omega t}\right]$

  $u(t)=u_0\cos(\omega t-\varphi) = \mathrm{Re}\left[u e^{i\varphi}e^{i\omega t}\right] = \mathrm{Re}\left[\tilde{u}(\omega)e^{i\omega t}\right]$
- Differenciating: $\dot{u}(t) = \mathrm{Re}\left[i\omega\cdot\tilde{u}(\omega)e^{i\omega t}\right]$

  $\ddot{u}(t) = \mathrm{Re}\left[-\omega^2\cdot\tilde{u}(\omega)e^{i\omega t}\right]$
- Substituting in the EOM: $M\ddot{u}(t) + R\dot{u}(t) + Ku(t) = F_{driv}\cos(\omega t)$

$$\tilde{u}(\omega) = \frac{F_{driv}}{(K-M\omega^2)+Ri\omega}$$

NOTE: This is the **particular** solution in complex form for a damped SDOF system. In Acoustics, most of the times, we are interested in the particular solution, which is the one not vanishing as time goes by.

NOTE: Differential equation became second order equation with time-harmonic ansats!

If the system is excited with $\omega_0^2=K/M$ → Resonance (dominated by damping)

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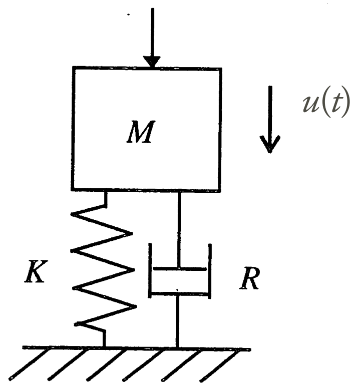
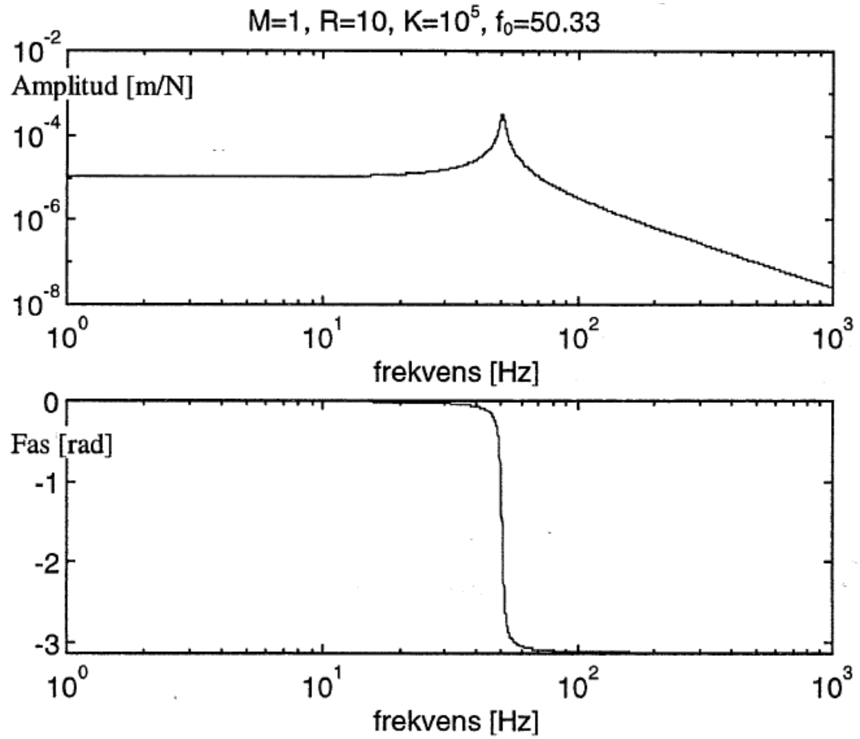
SDOF – Frequency response function

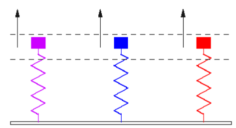
- Output / Input $$\frac{\tilde{u}(\omega)}{F_{driv}} = \frac{1}{(K - M\omega^2) + Ri\omega}$$

- The ratio is a complex number → transfer function

$F(t) = F_{driv} \cdot \cos(\omega t)$







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SDOF – Frequency response functions (FRF)

- In general, FRF = transfer function, i.e.:
  - Contains system information
  - Independent of outer conditions
  - **Frequency domain relationship between input and output of a linear time-invariant system**

$$H_{ij}(\omega) = \frac{\tilde{s}_i(\omega)}{\tilde{s}_j(\omega)} = \frac{\text{output}}{\text{input}}$$

- Different FRFs can be obtained depending on the measured quantity

Measured quantity	FRF	
Acceleration (a)	Accelerance = $N_{dyn}(\omega)$ = a/F	Dynamic Mass = $M_{dyn}(\omega)$ = F/a
Velocity (v)	Mobility/admitance = $Y(\omega)$ = v/F	Impedance = $Z(\omega)$ = F/v
Displacement (u)	Receptance/compliance = $C_{dyn}(\omega)$= u/F	Dynamic stiffness = $K_{dyn}(\omega)$ = F/u

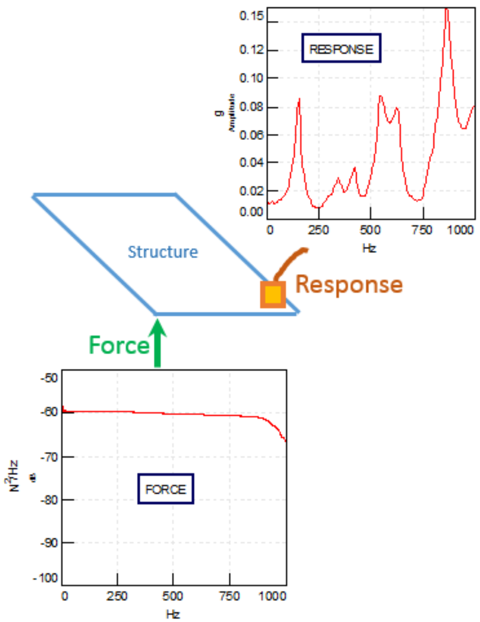
$$C_{dyn}(\omega) = \frac{\tilde{u}(\omega)}{F_{driv}(\omega)} = \frac{1}{(K - M\omega^2) + Ri\omega}$$

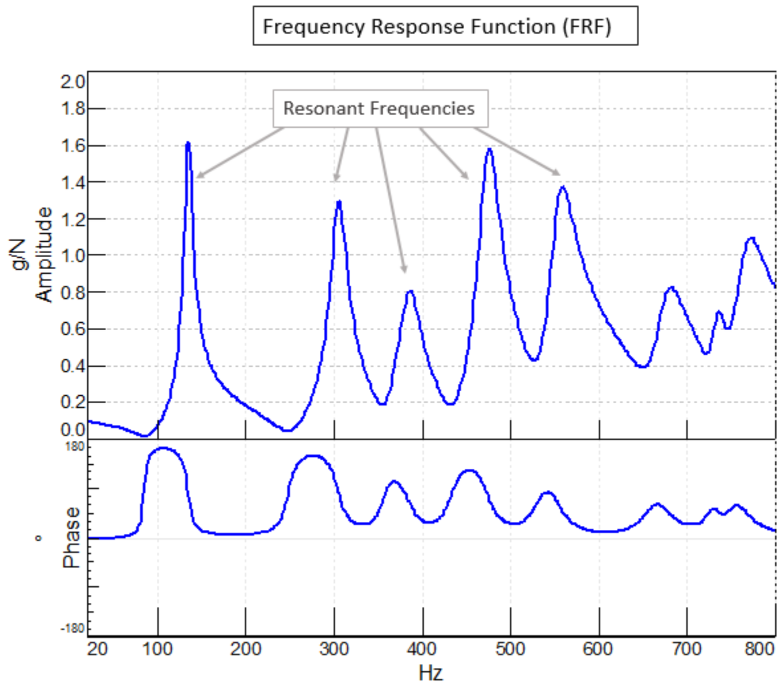
$$K_{dyn}(\omega) = C_{dyn}(\omega)^{-1} = -M\omega^2 + Ri\omega + K$$

FRF of a complex system

- What happens if we evaluate FRF of a complex (linear) system such as a plate?
    - We gain useful info on it!

$$H_{ij}(\omega) = \frac{\tilde{s}_i(\omega)}{\tilde{s}_j(\omega)} = \frac{\text{output}}{\text{input}}$$

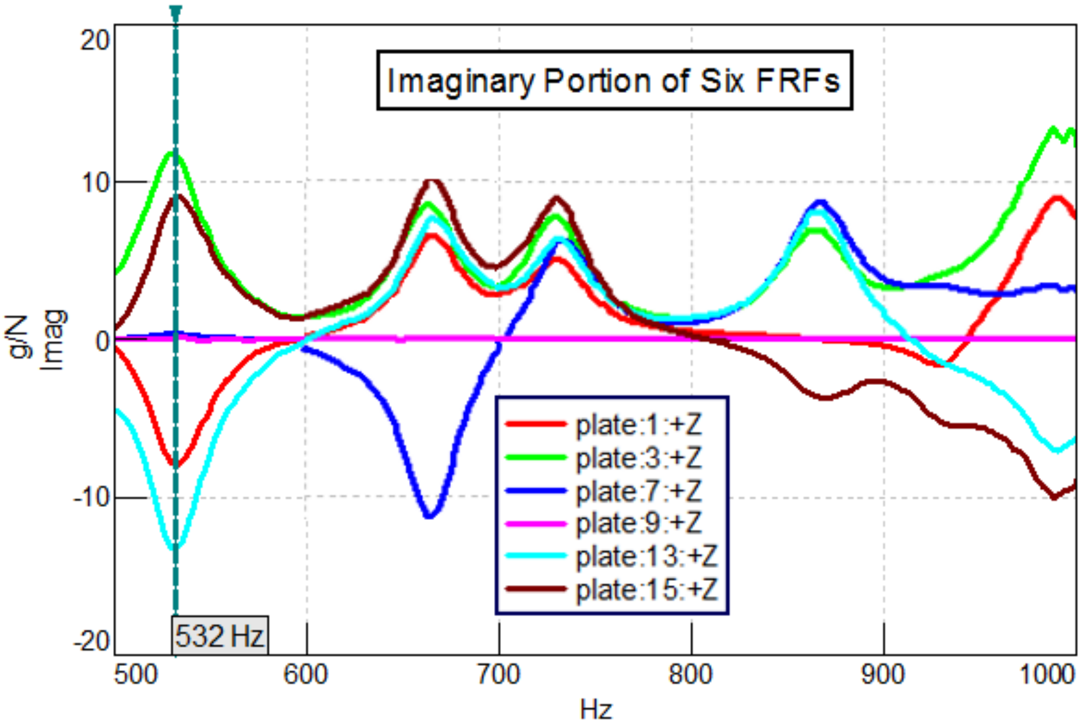


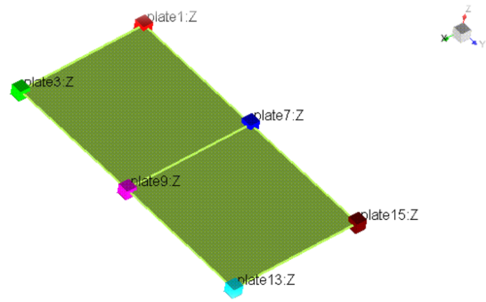


[https://community.sw.siemens.com/s/article/what-is-a-frequency-response-function-frf]

FRF of a complex system

- Real and imaginary parts –
    - the imaginary part has interesting information (Phase ϕ)





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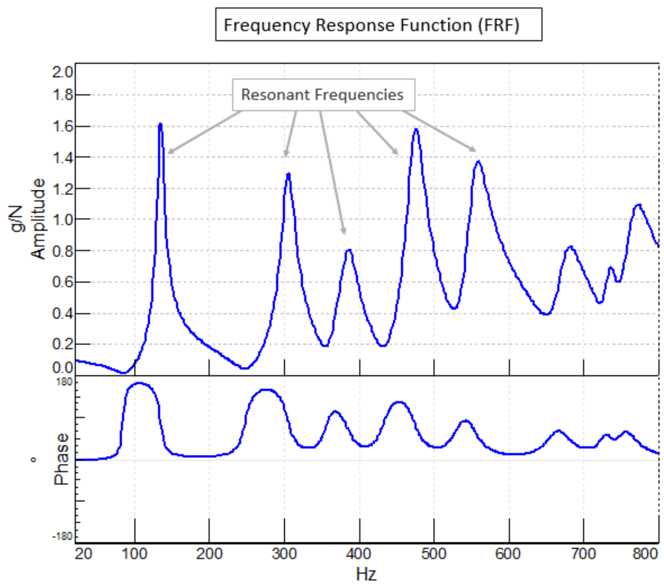
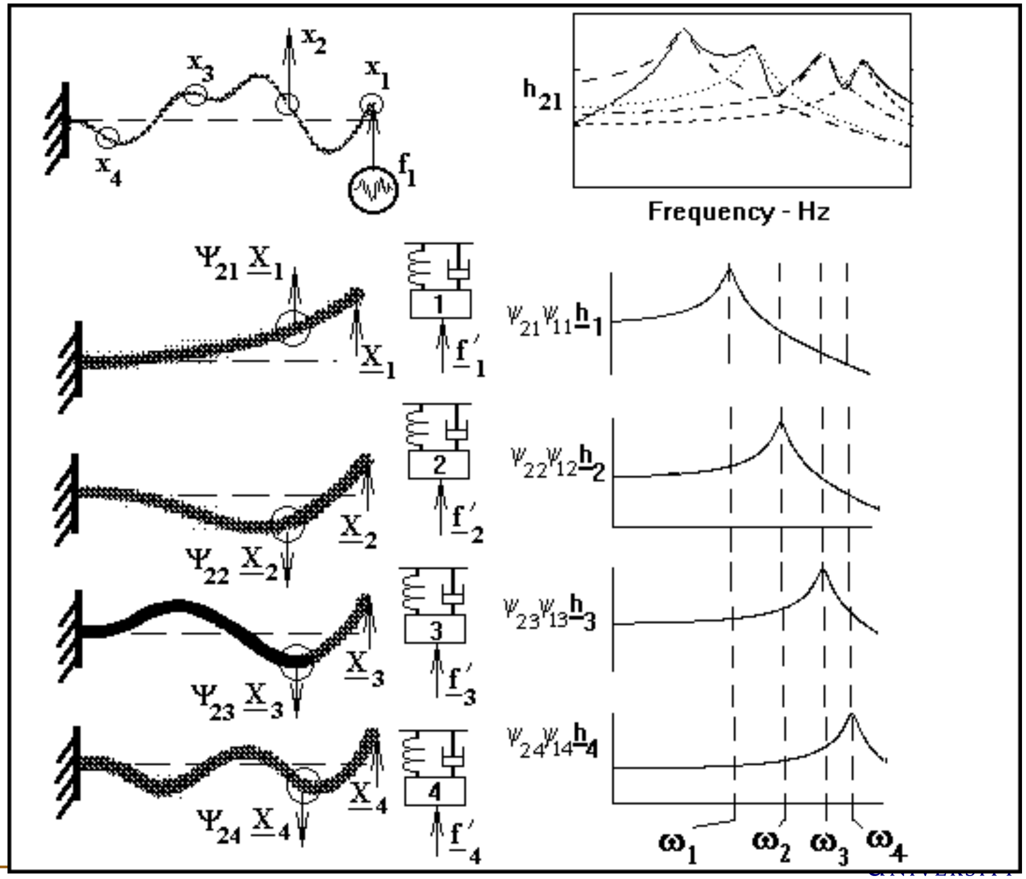
FRF of a complex system

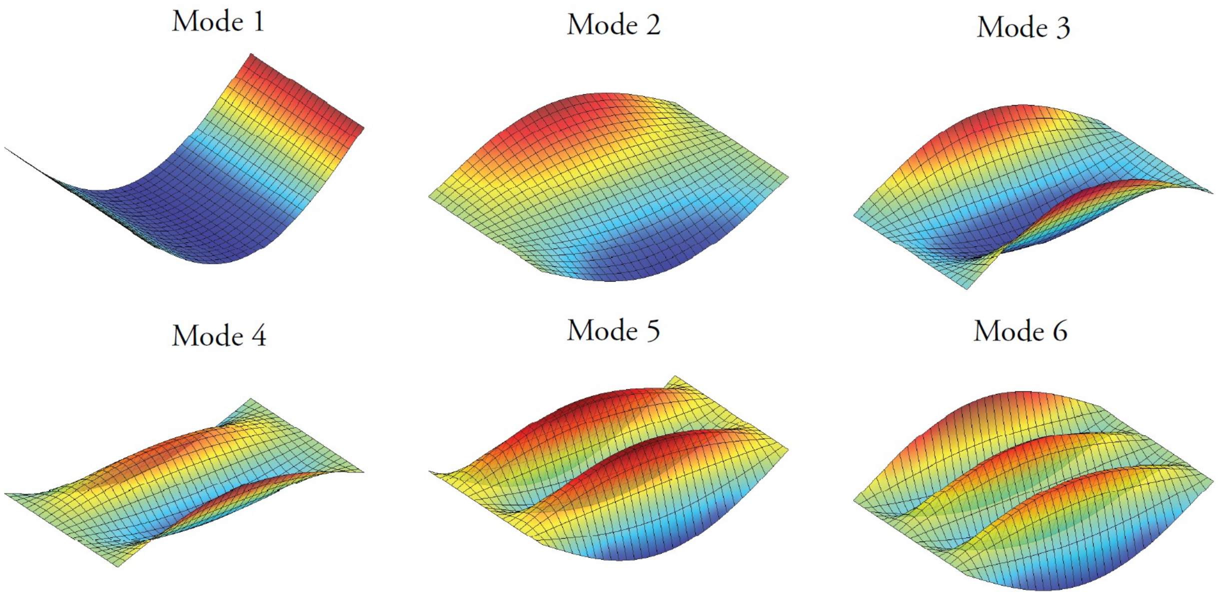
- Each peak is showing a natural frequency
  - Each peak is a mass-spring-damper SDOF system?!

- Modal superposition



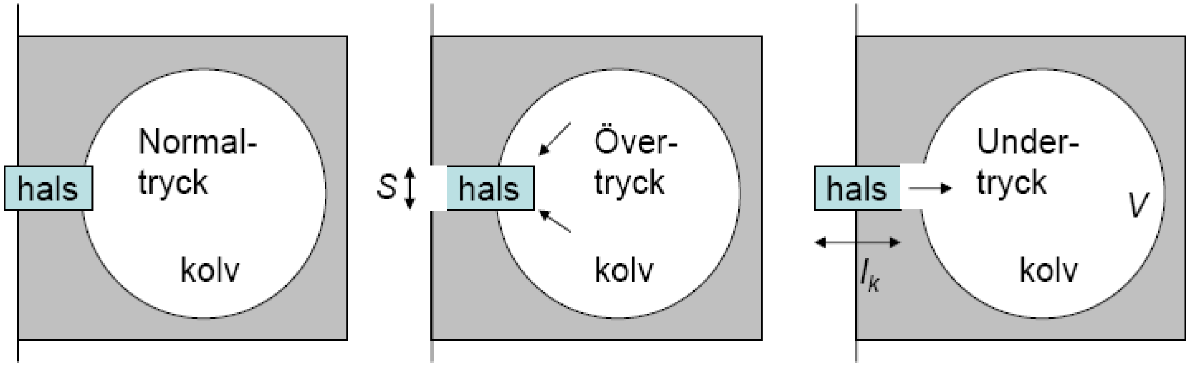


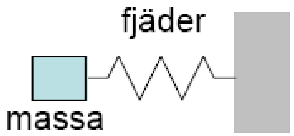
Mode shapes – Example floor



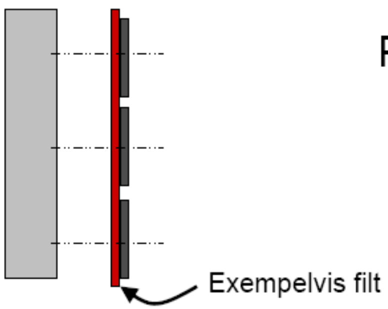
NOTE: In floor vibrations, modes are superimposed on one another to give the overall response of the system. Fortunately it is generally sufficient to consider only the first 3 or 4 modes, since the higher modes are quickly extinguished by damping.

Helmholtz resonator

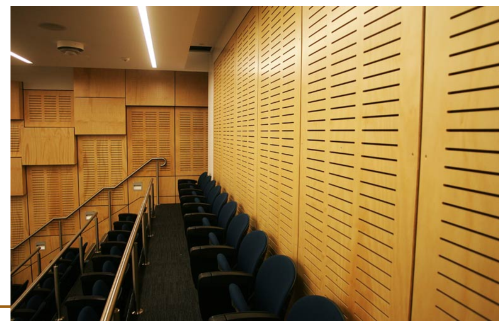




$$f_r = \frac{c}{2\pi}\sqrt{\frac{S}{l_k V}}$$



Perforerad absorbent.



Outline

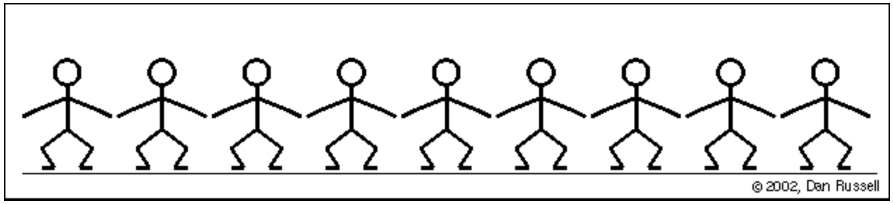
- Introduction
- Wave propagation in solid media
- Wave propagation in fluids
- Summary

Learning outcomes

- Wave propagation in solid media
  - Longitudinal/quasi-longitudinal waves
  - Shear waves
  - Bending waves
- Wave equation solution
- Wave propagation in fluids
- Wave phenomena
- Musical acoustics

Waves

- A wave is a “disturbance”, an oscillation that moves energy from one place to another.
- Only energy is transferred as a wave moves, not matter. The mass just moves up and down, or back and forth.
- The wave moves through a substance, the *medium*. That medium moves back and forth repeatedly, returning to its original position. But the wave travels along the medium.

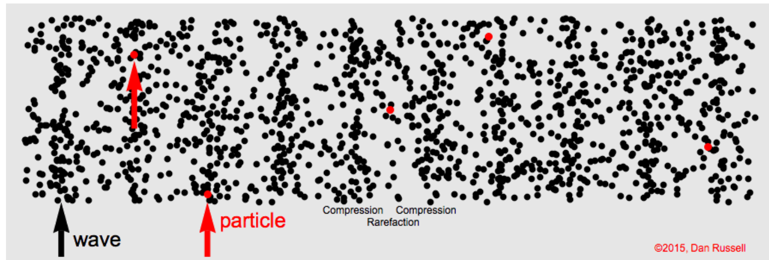


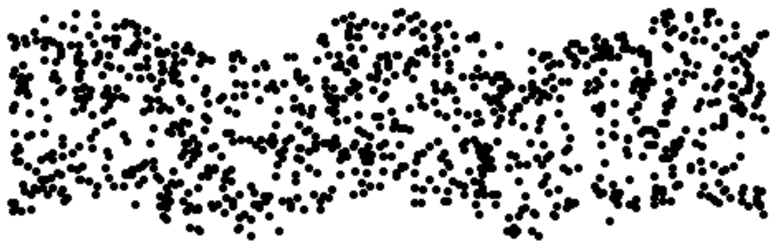
[https://www.acs.psu.edu/drussell/Demos/waves-intro/waves-intro.html]

Wave is a disturbance that travels in space!
People jumps up and sits down. None is carried away with the wave.

Types of waves – classification

- Depending on propagation media
  - Mechanical waves (solids and fluids)
  - Electromagnetical waves (vacuum)
- Propagation direction
  - 1D, 2D and 3D
- Based on periodicity
  - Periodic and non-periodic
- Based on particles' movement in relation with propagation direction:
  - Longitudinal waves (solids and fluids)
  - Transverse waves (solids)
- More?





NOTE: waves do not transport mass, just energy

Types of waves in solid media

- Longitudinal waves (∞ medium ≈ beams)
  - Quasi-longitudunal waves (finite ≈ plates)

$$\frac{\partial^2 u_x}{\partial x^2} - \frac{\rho}{E'}\frac{\partial^2 u_x}{\partial t^2} = 0$$

$$c_L = \sqrt{\frac{E}{\rho}}$$

$$c_{qL} = \sqrt{\frac{E'}{\rho}} = \sqrt{\frac{E}{\rho(1-\upsilon^2)}}$$

- Shear waves

$$\frac{\partial^2 u_y}{\partial x^2} - \frac{\rho}{G}\frac{\partial^2 u_y}{\partial t^2} = 0$$

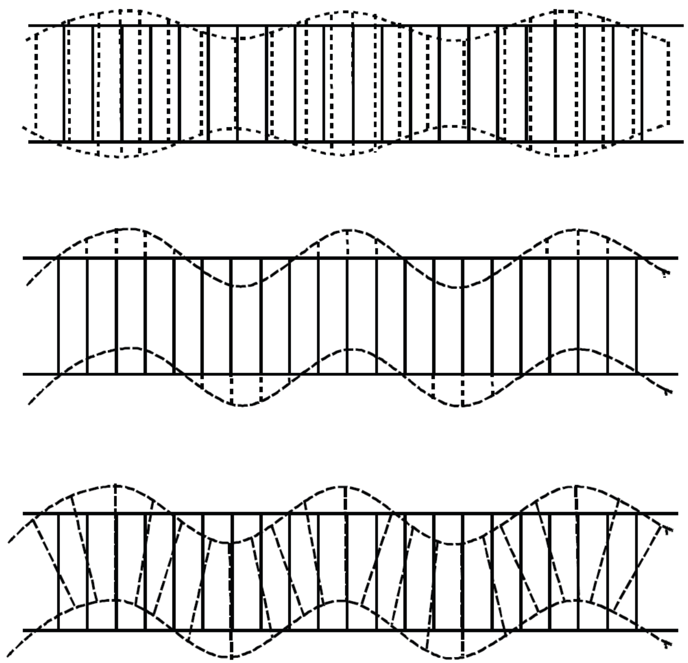
$$c_{sh} = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{E}{2(1+\upsilon)\rho}}$$

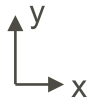
- Bending waves (flexural, dispersive)

$$B\frac{\partial^4 u_y}{\partial x^4} + m\frac{\partial^2 u_y}{\partial t^2} = 0$$

$$c_{B(\omega)} = \sqrt{\omega}\sqrt[4]{\frac{B}{m}}$$

Plate: E, G, ρ, υ, h





$$m = \rho h \qquad B_{beam} = E\frac{bh^3}{12}$$

$$B_{plate} = \frac{Eh^3}{12(1-\upsilon^2)}$$

NOTE: torsional waves (beams and columns) are not addressed here

Outline

Introduction

Wave propagation in solid media

Wave propagation in fluids

Summary

More types of waves in solid media

- Longitudinal waves
- Shear waves
- Torsional waves
- Bending waves
- Rayleigh waves
- Lamb waves
- …

Derivation of longitudinal wave equations (I)

- No details provided - Only for educational reasons
- General approach to derive equations of motion:

1. Newton's law – dynamic equilibrium
2. Constitutive relations – forces, stresses and strains
    - Relations between two physical quantities in a material

    a. Force – stress

    b. Stress – strain
3. Strain – displacement relation (definition)

General form of a wave equation

One dimension: $\frac{\partial^2 u_x}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2} = 0$

Three dimensions: $c^2\nabla^2 u - \ddot{u}=0$

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f \qquad \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Laplacian

https://en.wikipedia.org/wiki/Laplace_operator

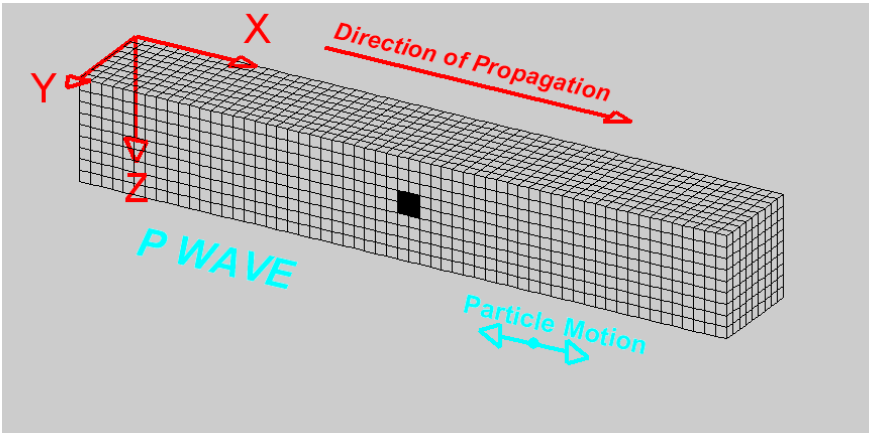
- It takes some physics reasoning and some math but we have an expression that we can use with most wave types that are relevant in acoustics and vibrations!
- Not however with the most important structural waves in acoustics, which is a bit special!

Waves in solid media

- Now we can go through some kinds of waves in solid media!

Longitudinal waves

- P waves (primary waves in seismology)



http://www.geo.mtu.edu/UPSeis/waves.html

Longitudinal waves

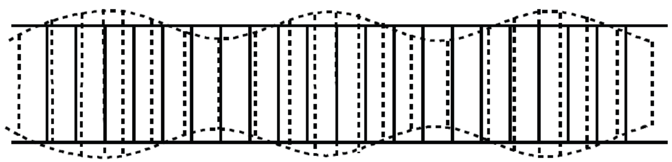
- Longitudinal waves (∞ medium ≈ beams)
  - Quasi-longitudunal waves (finite ≈ plates)

$$\frac{\partial^2 u_x}{\partial x^2} - \frac{\rho}{E'}\frac{\partial^2 u_x}{\partial t^2} = 0$$

$$c_L = \sqrt{\frac{E}{\rho}}$$

$$c_{qL} = \sqrt{\frac{E'}{\rho}} = \sqrt{\frac{E}{\rho(1-\upsilon^2)}}$$

Plate: E, G, ρ, υ, h

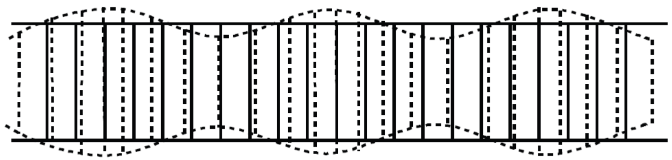


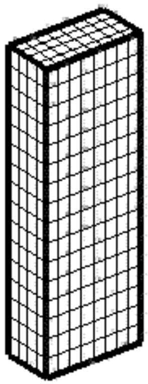
- Longitudinal wave, is the one where the points in the wave medium oscillate in the direction of propagation. So the medium is compressed, and the restoring force is given by pressure. *An example is a sound wave or a regular spring.*
- The opposite is a transverse wave where the points in the wave medium pivot perpendicular to the direction of propagation. *Examples: stringed instruments, the water in a pond and electromagnetic radiation.*

Longitudinal waves

- Longitudinal waves (∞ medium ≈ beams)
  - Quasi-longitudunal waves (finite ≈ plates)

Plate: E, G, ρ, υ, h

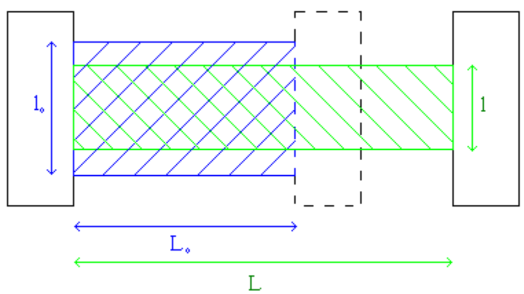




  - Poisson's ratio, the number or the transverse contraction. It is a material constant showing how a material reacts to compressive and tensile forces. When a material (blue) is stretched in one direction, it contracts in other directions (green).

$$c_L = \sqrt{\frac{E}{\rho}}$$

$$c_{qL} = \sqrt{\frac{E'}{\rho}} = \sqrt{\frac{E}{\rho(1-\upsilon^2)}}$$

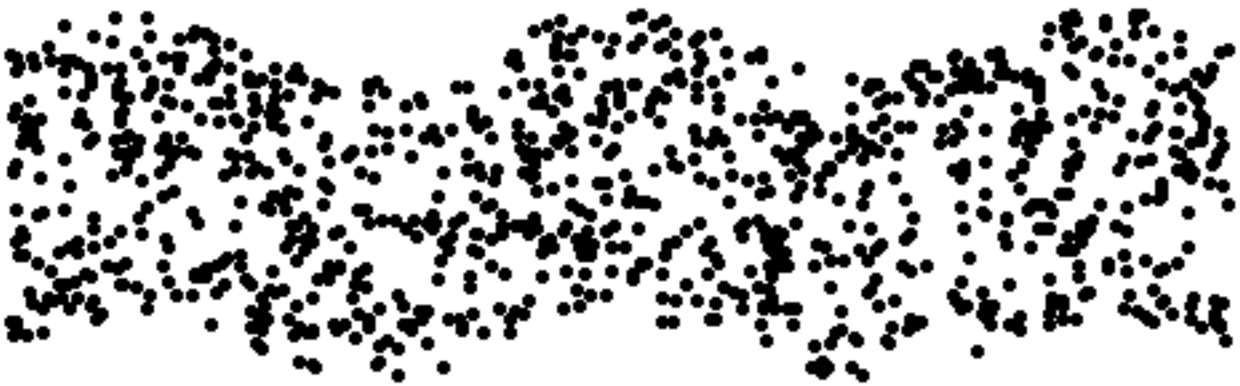


$$\nu = -\frac{\text{Strain in direction of load}}{\text{Strain at right angle to load}}$$

$$\nu = -\frac{\varepsilon_{lateral}}{\varepsilon_{axial}}$$

Shear waves

- Shear waves / transverse waves



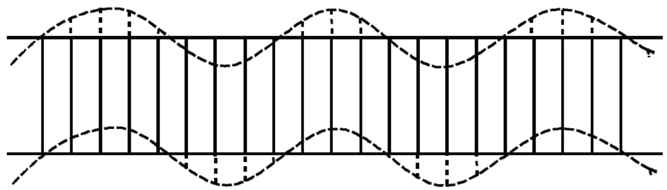
Shear waves

- Shear waves

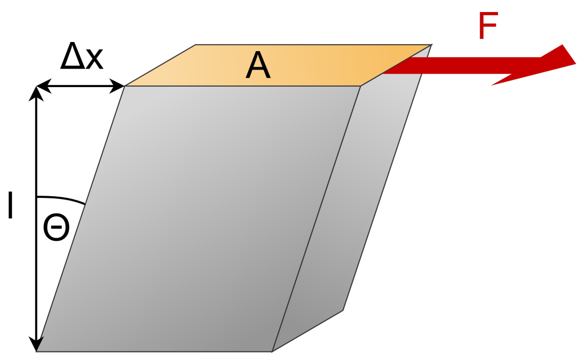
Plate: E, G, ρ, υ, h

$$\frac{\partial^2 u_y}{\partial x^2} - \frac{\rho}{G}\frac{\partial^2 u_y}{\partial t^2} = 0$$

$$c_{sh} = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{E}{2(1+\upsilon)\rho}}$$



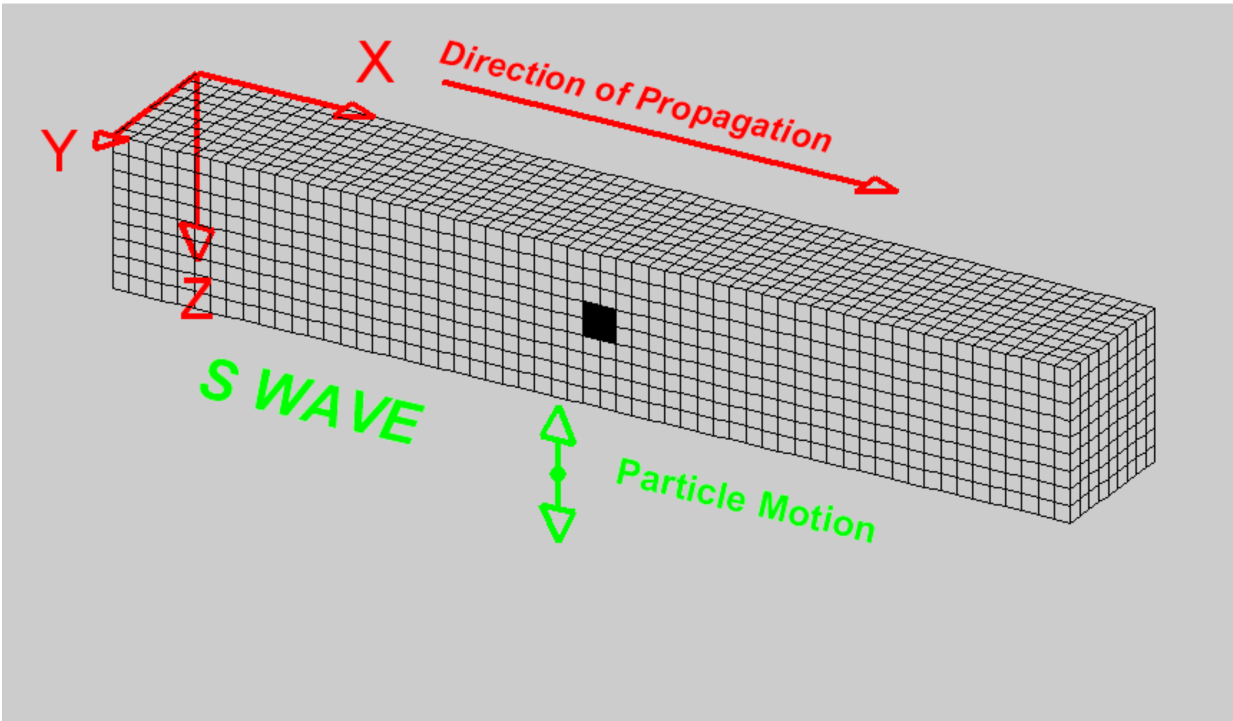
Shear, or shear strain, is a deformation without volume change. It is defined as the angular change created by the deformation.



Source: Wikipedia.se

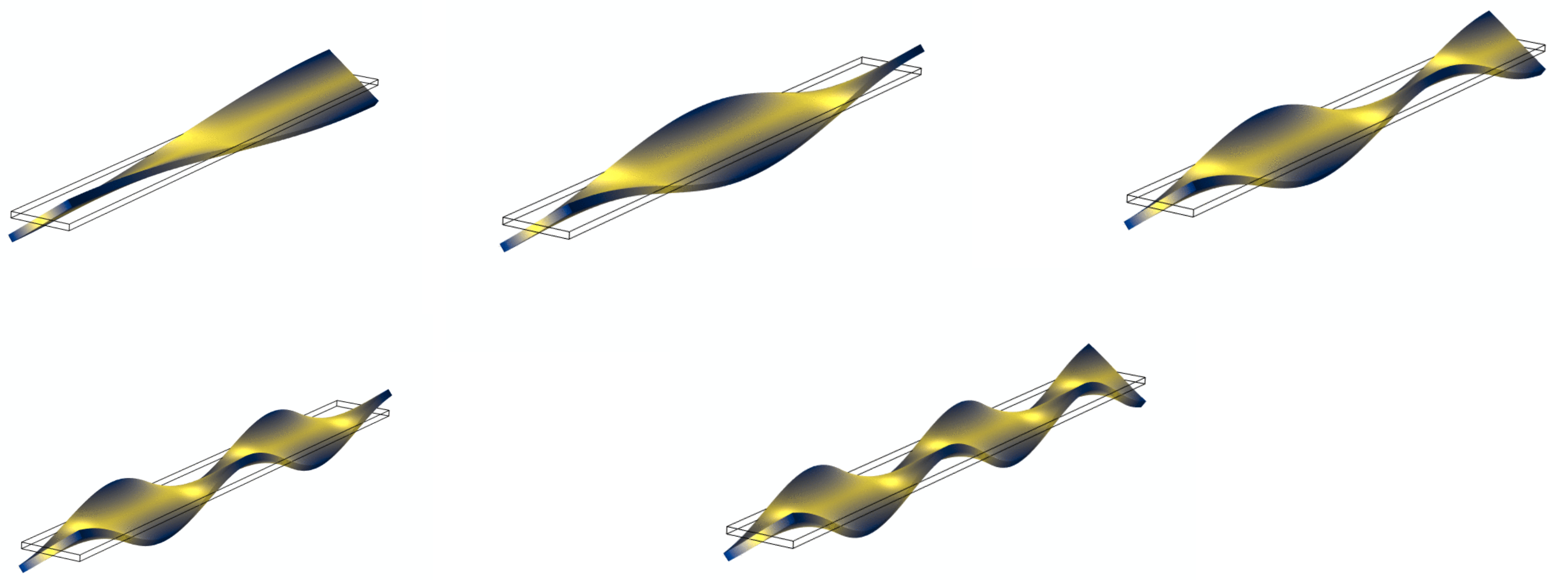
Shear waves

- S waves



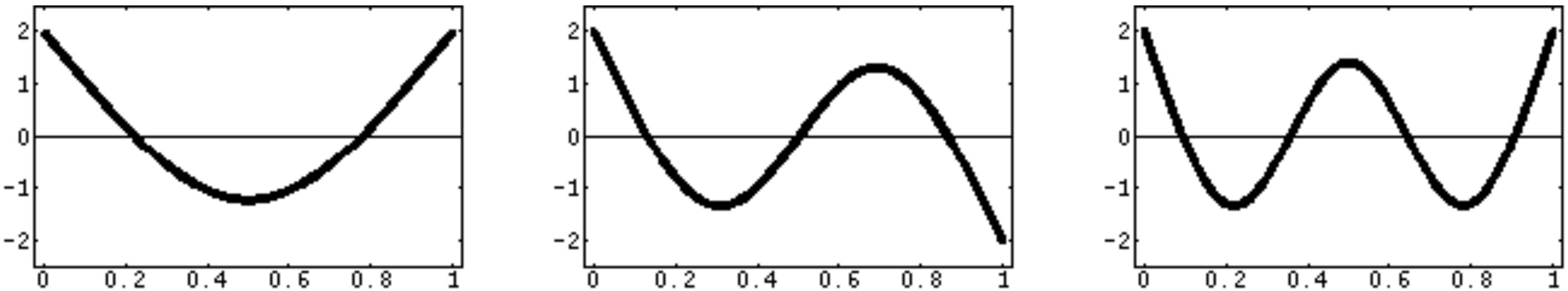
Torsional waves

- Torsional waves $$\frac{\partial^2\theta}{\partial x^2} - \frac{1}{c_T^2}\frac{\partial^2\theta}{\partial t^2} = 0 \quad \text{with} \quad c_T = \sqrt{\frac{GK}{\rho I_p}}$$



Bending waves

- **Bending waves** (free-free) – Böjvågor på svenska

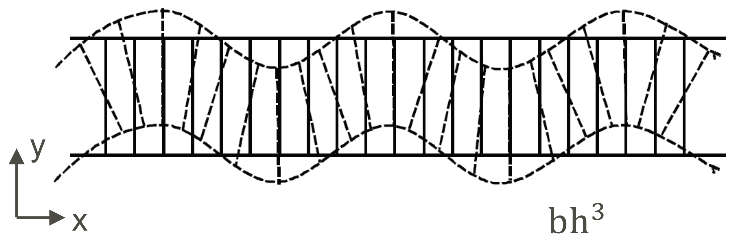


Bending waves

- Bending (or flexural) waves (dispersive)

$$B\frac{\partial^4 u_y}{\partial x^4} + m\frac{\partial^2 u_y}{\partial t^2} = 0 \qquad c_{B(\omega)} = \sqrt{\omega}\sqrt[4]{\frac{B}{m}}$$

Plate: E, G, ρ, υ, h



$$m = \rho h \qquad B_{beam} = E\frac{bh^3}{12}$$

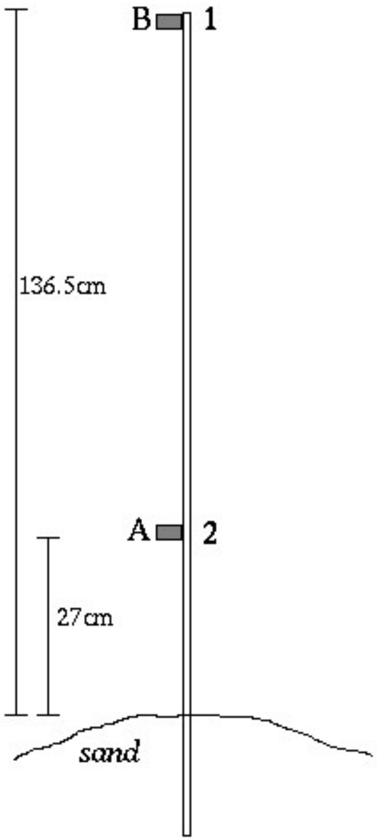
$$B_{plate} = \frac{Eh^3}{12(1-\upsilon^2)}$$

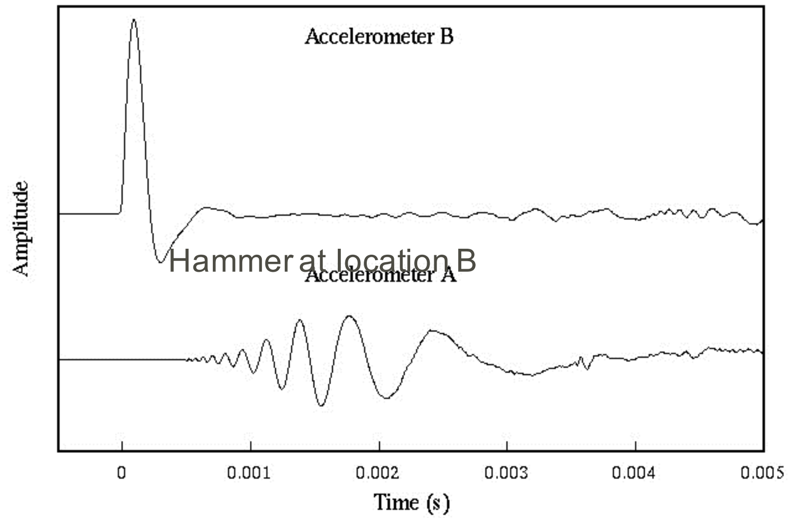
- Planar section remain plane
- Towards the middle line perpendicular cross-section remains perpendicular to the middle line after deformation (that is, shear deformation is neglected).
- Different form of wave equation!
- Happen when a transverse load is applied to a structure (beam, plate).
  - Examples?!
- Perhaps the most important structural wave in acoustics.
  - Next lecture F4 we will talk again about bending waves and explain why this is true. Spoiler: bending waves radiate sound well.

Bending waves

$$B\frac{\partial^4 u_y}{\partial x^4} + m\frac{\partial^2 u_y}{\partial t^2} = 0$$

$$c_{B(\omega)} = \sqrt{\omega}\sqrt[4]{\frac{B}{m}}$$





- Force pulse is very clean at location B
- Pulse disperses by the time it reaches location A --- higher frequency waves travel faster and arrive first --- lower frequency waves travel slower and arrive later

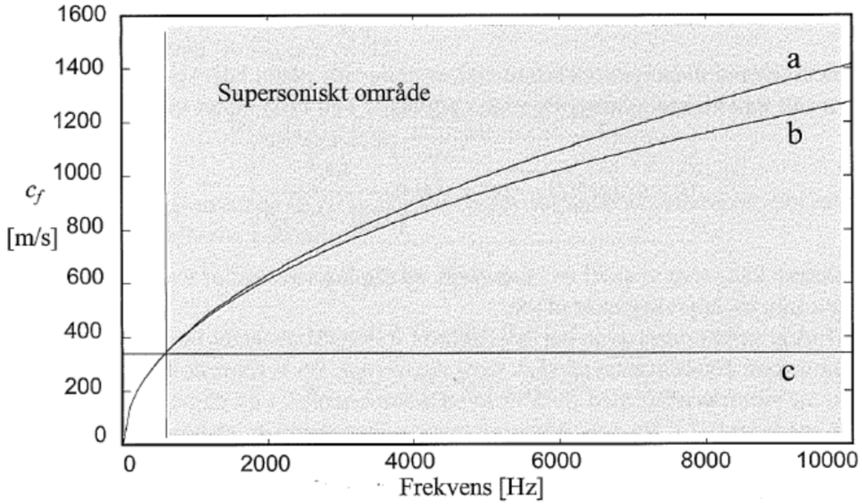
https://www.acs.psu.edu/drussell/Demos/Dispersion/Flexural.html

Coincidence

- Relation between different wavespeeds

$$c_{B(\omega)} = \sqrt{\omega}\sqrt[4]{\frac{B}{m}}$$

- Dispersion relations – frequency dependence of wave speed



Figur 6-28 Fashastigheten för en cirkulärcylindrisk stålbalk med diametern 5 cm. a) Enligt Bernoulli-Eulerteori, b) enligt Timoshenkoteori. c) Fashastigheten för en kompressionsvåg i luft. Den frekvens där böjvågens fashastighet är lika med ljudhastigheten i det omgivande mediet kallas koincidensfrekvens.

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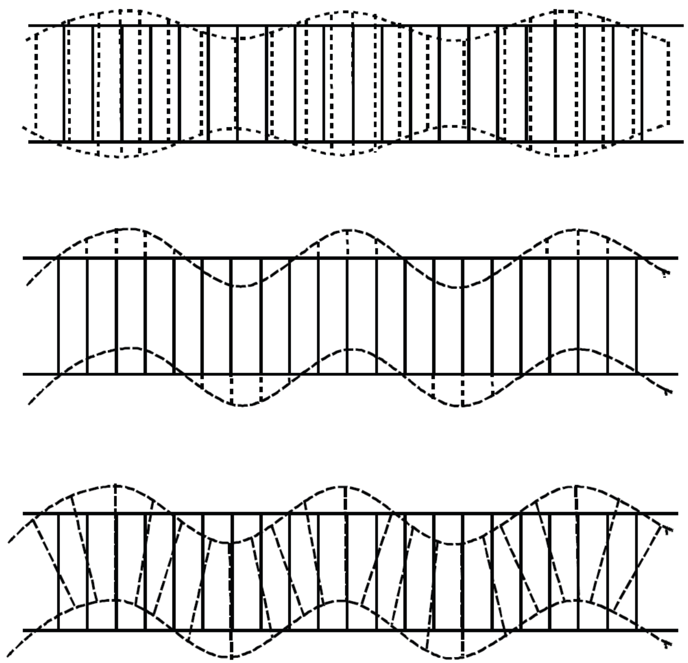
Summary - Types of waves in solid media

- Longitudinal waves (∞ medium ≈ beams)
  - Quasi-longitudunal waves (finite ≈ plates)

Plate: E, G, ρ, υ, h



$$\frac{\partial^2 u_x}{\partial x^2} - \frac{\rho}{E'}\frac{\partial^2 u_x}{\partial t^2} = 0$$

$$c_L = \sqrt{\frac{E}{\rho}}$$

$$c_{qL} = \sqrt{\frac{E'}{\rho}} = \sqrt{\frac{E}{\rho(1-\upsilon^2)}}$$

- Shear waves

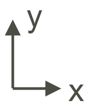
$$\frac{\partial^2 u_y}{\partial x^2} - \frac{\rho}{G}\frac{\partial^2 u_y}{\partial t^2} = 0$$

$$c_{sh} = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{E}{2(1+\upsilon)\rho}}$$

- Bending waves (dispersive)

$$B\frac{\partial^4 u_y}{\partial x^4} + m\frac{\partial^2 u_y}{\partial t^2} = 0$$

$$c_{B(\omega)} = \sqrt{\omega}\sqrt[4]{\frac{B}{m}}$$



$$m = \rho h \qquad B_{beam} = E\frac{bh^3}{12}$$

$$B_{plate} = \frac{Eh^3}{12(1-\upsilon^2)}$$

- Torsional waves

Solution to the wave equation

One dimension: $\frac{\partial^2 u_x}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2} = 0$

- What is the simplest shape of a sound wave?
  - An harmonic shape of sinusoidal shape
    - » $u = A \sin a(x - ct)$; $u = A \cos a(x - ct)$
  - It follows that $u = Ae^{\pm ia(x-ct)}$ is also a solution.
  - As $u = A \ln a(x - ct)$; or $u = A\sqrt{a(x - ct)}$, which are not oscillatory.
- It turns out that any function of the form $u = f(x - ct)$ is solution.
- No assumption is made on f – except on its argument.
  - What does that imply?

Solution to the wave equation

$$\text{One dimension: } \frac{\partial^2 u_x}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2} = 0$$

- A wave is translated, unchanged in shape, along x (space)
- A given point on a wave is translated unchanged with speed *c*.
- This operation defines propagation!

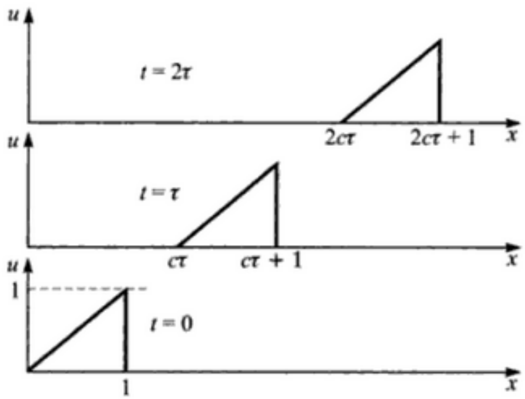
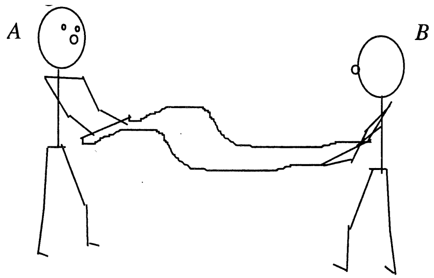


Figure 1.1 Sketches showing the waveform in space when the solution is a section of a ramp function.

[Blackstock, Fundamentals of physical acoustics]

Solution to the wave equation



- Check p.25 of the compendium

Vi kan tänka oss två personer som skakar en matta, vi kallar dem A och B. Vi tänker oss vidare att B håller sin kant stilla medan A gör en plötslig rörelse uppåt vid sin kant. Resten av mattan vill nu följa med i denna rörelse, med början med de punkter på mattan som är närmast A. Rörelsen fortsätter sedan att sprida sig med en konstant hastighet tills den når B. Om A fortsätter att skaka sin ände upp och ned, och provar olika takt i skakandet, olika frekvens, så kommer de finna att om man skakar snabbt så blir ”pulsen”, eller våglängden, kort och om man skakar långsamt så blir våglängden lång. Men oavsett vilken frekvens de skakar med så kommer spridningshastigheten att vara densamma. Med andra ord, händelsen att ”röra sig uppåt” sprider sig längs mattan med en viss hastighet som vi kan kalla c, vågutbredningshastighet. Det känns naturligt att c beror på mattans vikt och hur hårt A och B drar i mattan, hur stor spänningen är. Är massan stor, tung matta, transporteras vågen långsamt. Är spänningen stor går vågen snabbt. Den initiala förskjutningen kommer att repeteras vid en punkt belägen en sträcka x från A, och detta sker efter x/c sekunder, det vill säga den tid det tar för vågen att utbreda sig sträckan x.

Det är viktigt att inse att ingen massa transporteras av vågen, vad som transporteras är endast möjligheten till rörelse. Massan i mattan rör sig endast upp och sedan ner igen.

I exemplet med mattan ovan var förskjutningen uppåt medan vågutbredningen går mellan A och B,

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Solution to the wave equation

$$\text{One dimension: } \frac{\partial^2 u_x}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2} = 0$$

- It turns out that any function of the form $u = f(x - ct)$ is solution.
- Prove it!
- Lead:

$$u(x,t) = u_+(t - x/c) \Rightarrow$$

$$\frac{\partial u}{\partial x} = -\frac{1}{c}u'_+(t - x/c) \quad ; \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}u''_+(t - x/c)$$

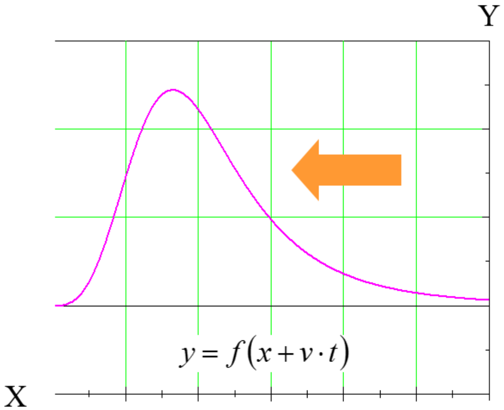
$$\frac{\partial u}{\partial t} = u'_+(t - x/c) \quad ; \quad \frac{\partial^2 u}{\partial t^2} = u''_+(t - x/c)$$

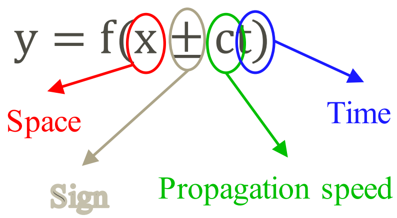
What is a wave then?

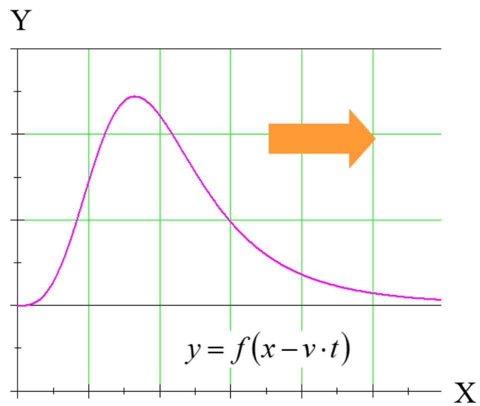
- A disturbance or deviation from a pre-existing condition. Its motion constitutes a transfer of information from one point in space to another.
- Time plays a key role – static displacement of a rubber band is a disturbance but not a wave.
  - Wave travels at finite speed (hitting a perfectly rigid rod making the rod moving as a unit is no wave, just rigid body motion)
  - The rod is elastic, the impulse travels from one end to the other.
- All mechanical waves travel in a material medium
- Many waves satisfy $c^2\nabla^2 u - \ddot{u}=0$ – but not all!

Wave equation solution

- Most general solution is forward and backward travelling wave.
  - d'Alambert's solution





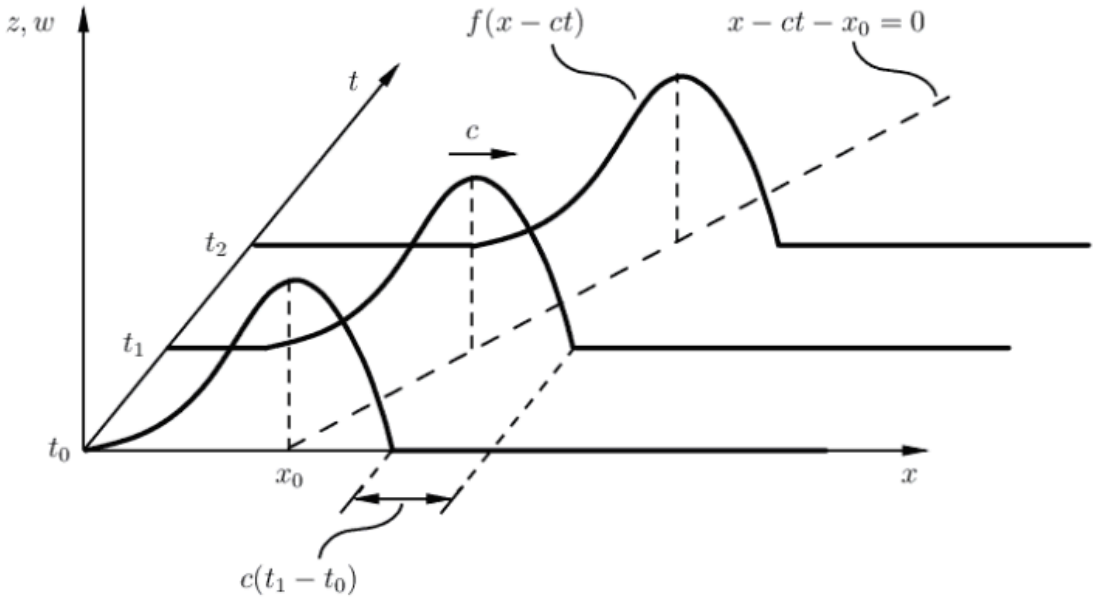


$$y = f\left(x \pm \frac{\omega}{k} t\right) = f\left(\frac{kx \pm \omega t}{k}\right) = f(kx \pm \omega t)$$

- Alternative forms:

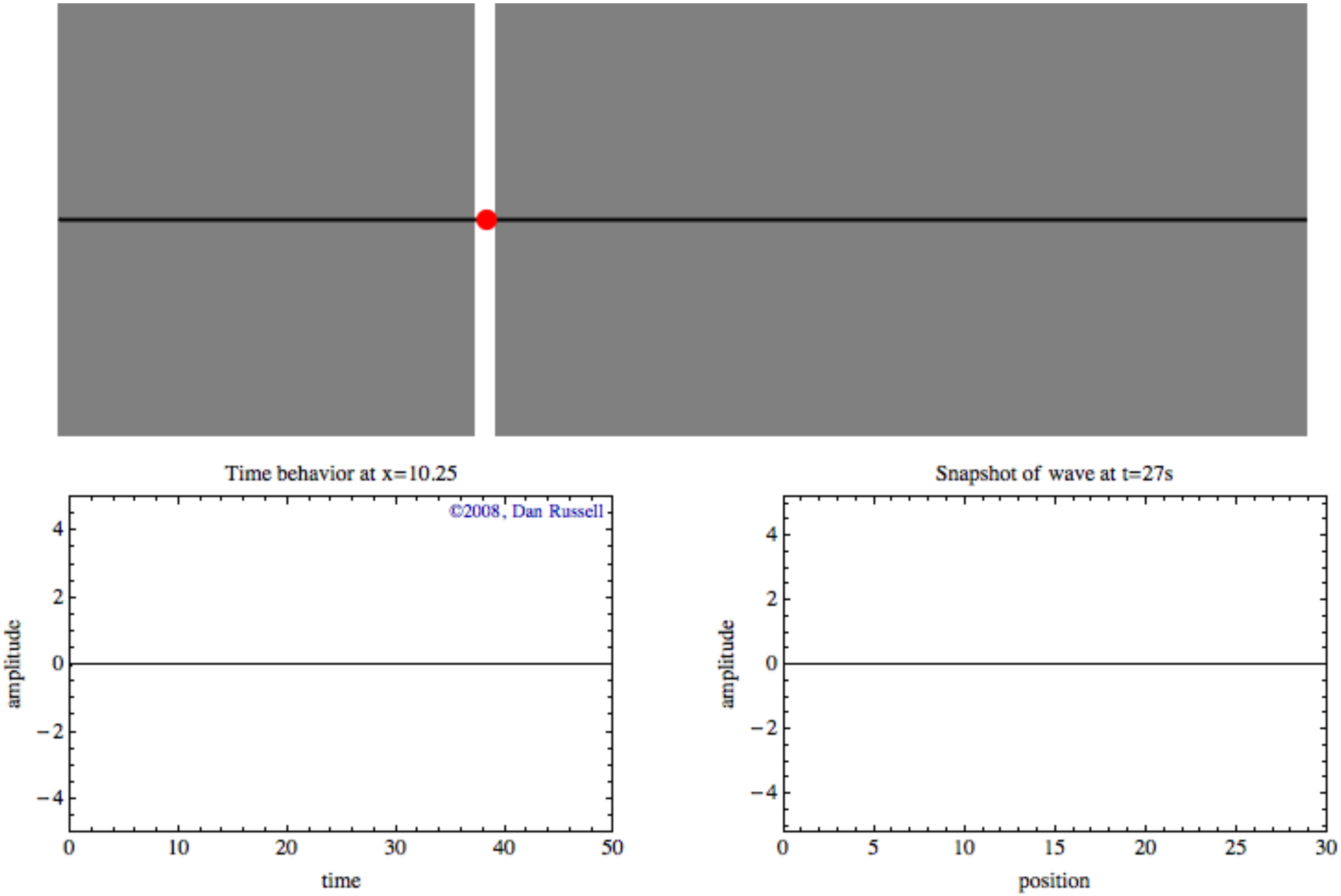
Wave equation solution

Travelling waves – d'Alambert's solution:



Wave equation solution

Time and position dependency: $u(x,t) = \widehat{u_+}\cos(\omega t - kx) = \widehat{u_+}e^{-i(\omega t - kx)}$



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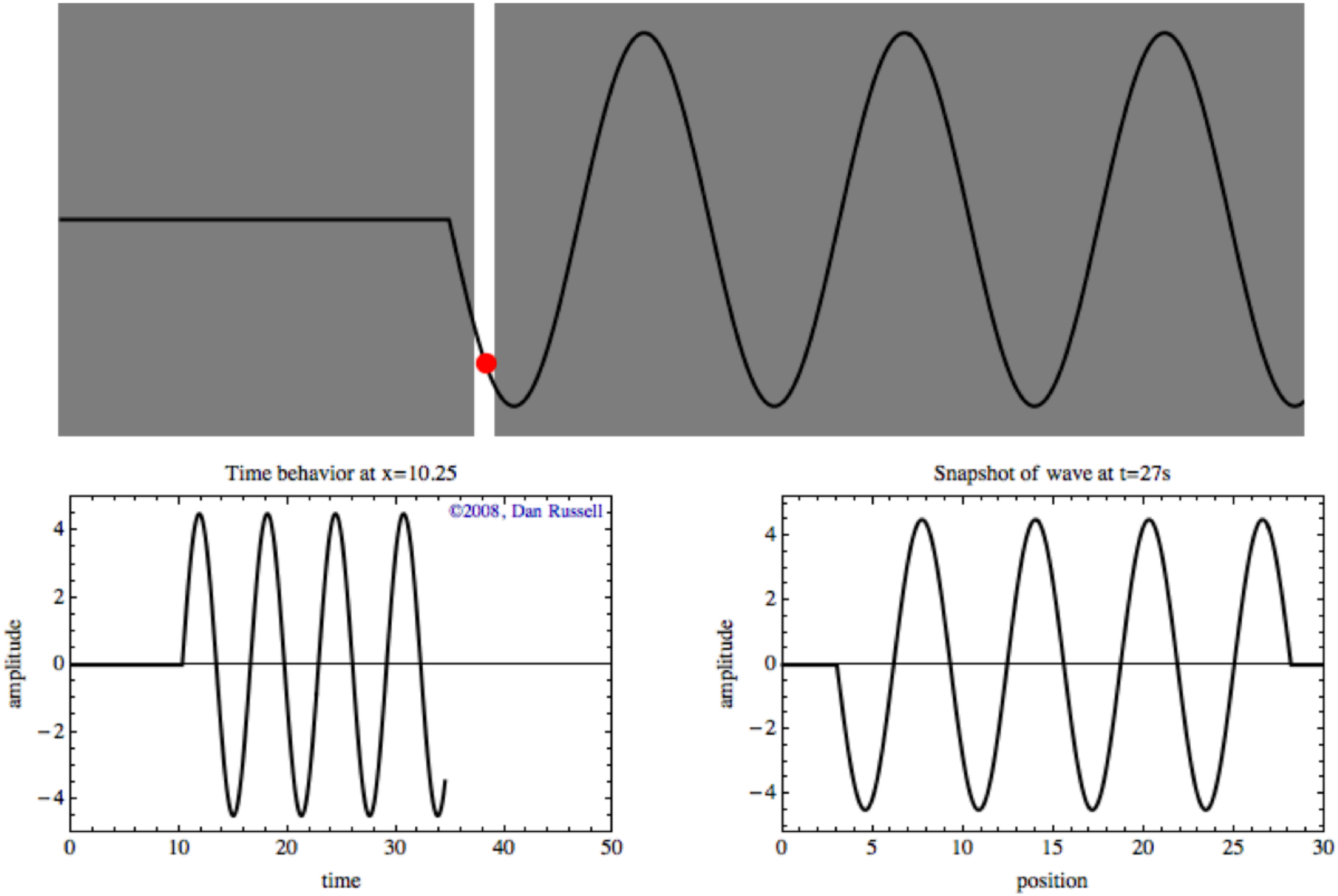
Wave equation solution (II)

[Still frame] Time and position: $u(x,t) = \widehat{u_+}\cos(\omega t - kx) = \widehat{u_+}e^{-i(\omega t - kx)}$



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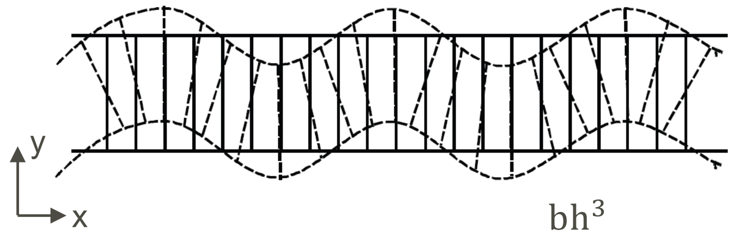
Bending waves – reprise

- Back to bending waves to say two things:
  - One about solution to bending waves
  - One about waves in general
- Bending waves (dispersive)

$$B\frac{\partial^4 u_y}{\partial x^4} + m\frac{\partial^2 u_y}{\partial t^2} = 0 \qquad c_{B(\omega)} = \sqrt{\omega}\sqrt[4]{\frac{B}{m}}$$

Plate: E, G, ρ, υ, h



$$m = \rho h \qquad B_{beam} = E\frac{bh^3}{12}$$

$$B_{plate} = \frac{Eh^3}{12(1-\upsilon^2)}$$

- Different form of wave equation!

Bending waves - solution

$$B\frac{\partial^4 u_y}{\partial x^4} + m\frac{\partial^2 u_y}{\partial t^2} = 0$$

- Due to the different form of equations with four-times spatial derivatives, the solution is more complex solutions including *near-field* terms

$$\zeta(x,t) = \hat{\zeta} e^{i(\omega t - k_B x)}. \qquad \zeta(x,t) = (\mathbf{A}e^{ik_B x} + \mathbf{B}e^{-ik_B x} + \mathbf{C}e^{k_B x} + \mathbf{D}e^{-k_B x})e^{i\omega t},$$

- Why near field!?
  - Because they decay rather quickly away from the *boundary* or *load point!*

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Bending waves - solution

- More complex solutions including near-field terms

$$\zeta(x,t) = \hat{\zeta} e^{i(\omega t - k_B x)}. \qquad \zeta(x,t) = (\mathbf{A}e^{ik_B x} + \mathbf{B}e^{-ik_B x} + \mathbf{C}e^{k_B x} + \mathbf{D}e^{-k_B x})e^{i\omega t},$$

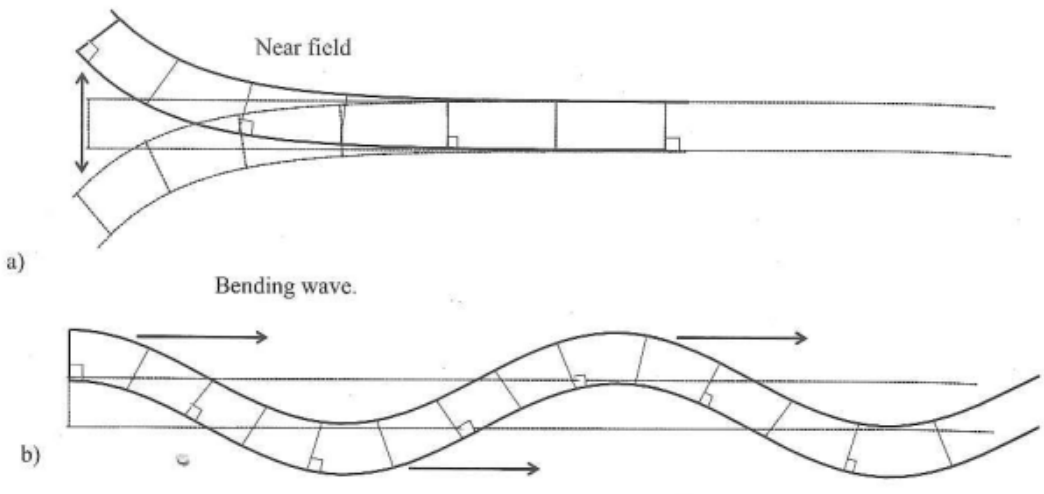
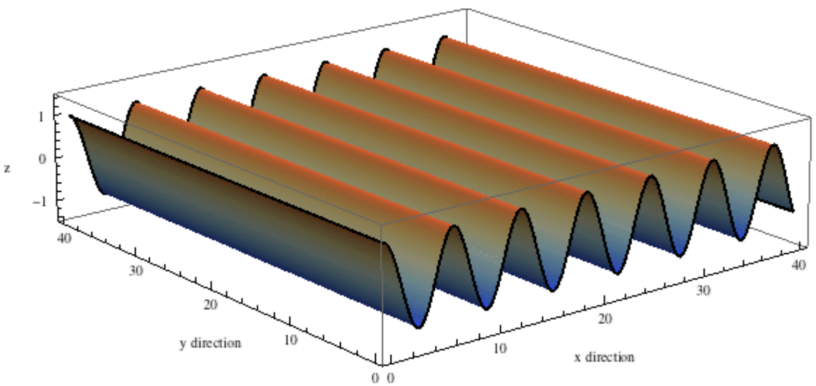


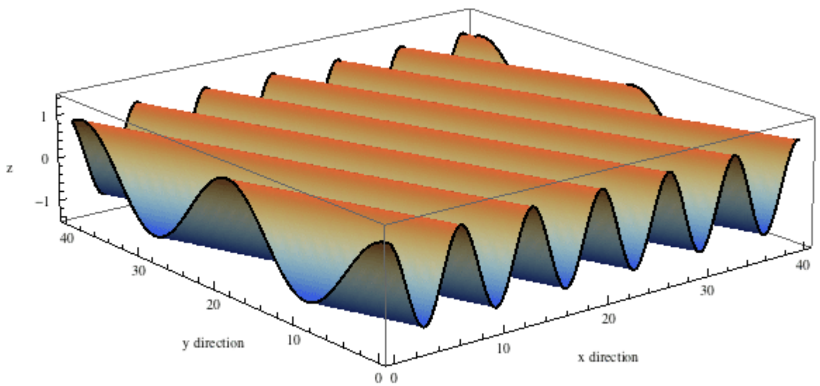
Figure 6-23 Solutions to the bending wave equation near the end of a beam. a) The near field is characterized by an amplitude that decays exponentially with distance from the excitation point. A reasonable engineering approximation would be to ignore the near field at distances greater than 1/3 of the bending wavelength. The near field is significant near boundaries, force application points, and other discontinuities. b) The bending wave can, if the losses are small, spread over large distances.

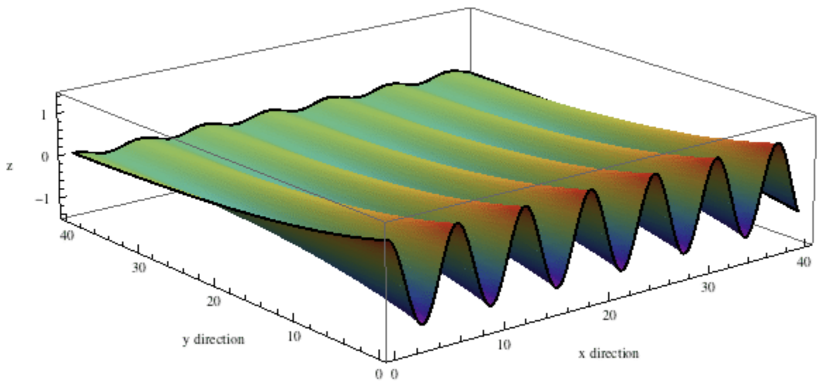
[*Sound and Vibration*, Wallin, Carlsson, Åbom, Bodén, Glav.]

Propagating waves VS Evanescent waves

- Waves may thus propagate or not propagate!
- Evanescent waves: decaying exponentially in one direction



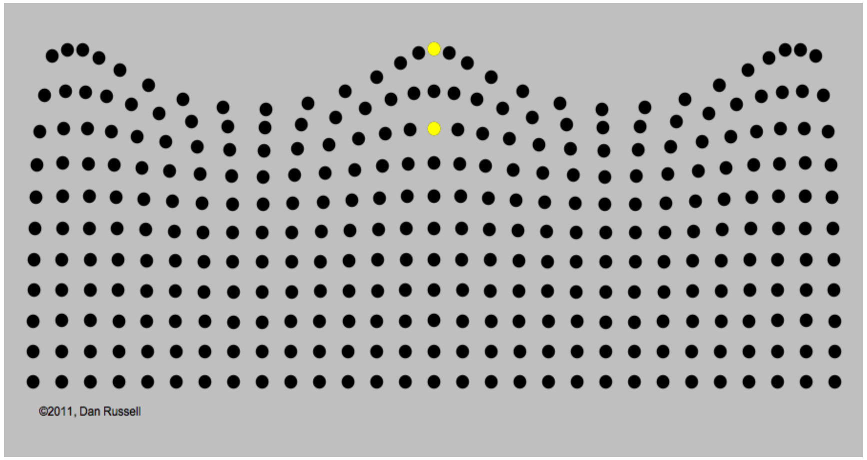


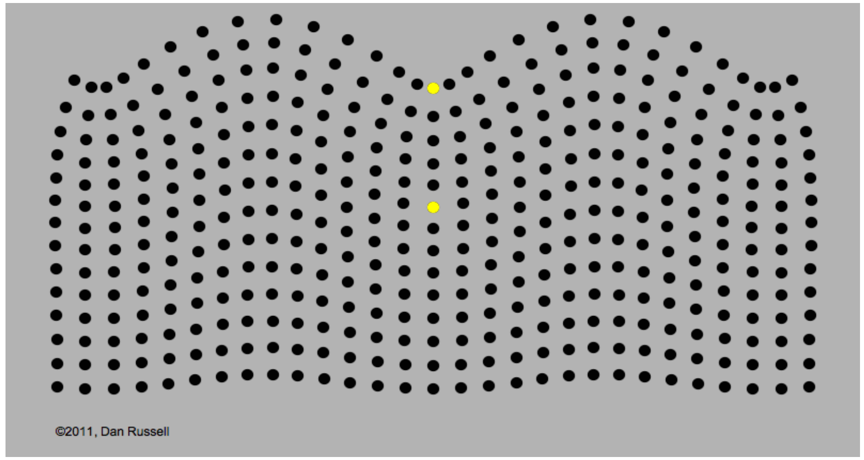


[https://www.acs.psu.edu/drussell/Demos/EvanescentWaves/EvanescentWaves.html]

Other types of waves…

- In reality, combinations of aforementioned waves can exist, e.g.





- <u>Surface waves</u>

Water waves

(long+transverse waves)

Particles in *clockwise circles*. The radius of the circles decreases increasing depth

Pure shear waves don't exist in fluids

- <u>Body waves</u>

Rayleigh waves

(long+transverse waves)

Particles in elliptical *paths*. Ellipses width decreases with increasing depth

Change from depth>1/5 of λ

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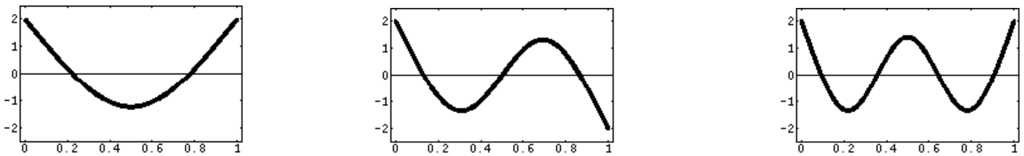
Boundary and initial conditions

- A structure can be mathematically described by
  - Equations of motion
  - Boundary conditions (in space)
  - Initial conditions (in time – irrelevant if harmonic motion $e^{i\omega t}$ is assumed)
- Boundary-value problem
- Examples in calculations for FEM (Finite Element Modeling)

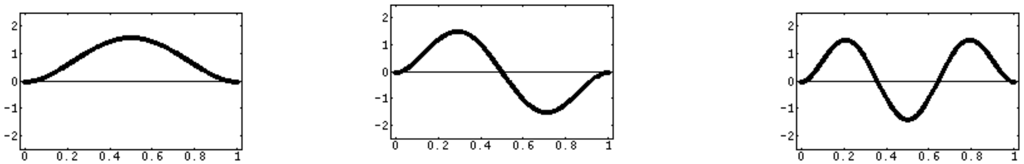
Boundary conditions (beam in bending)

- Structures and the waves in them behave differently depending on boundary conditions – i.e. How structures are connected at their ends

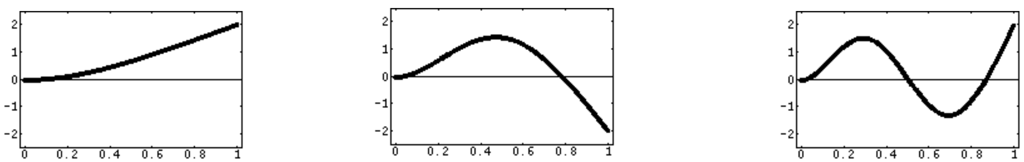
Free-free



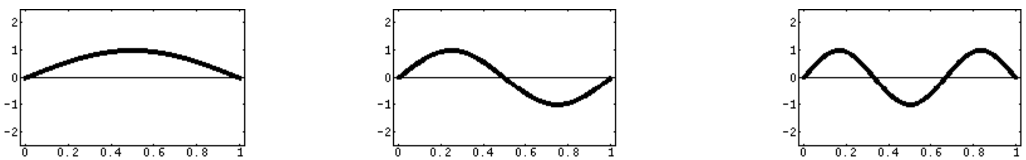
Clamped-clamped



Clamped-free



Simply supported both ends



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Summary

- Wave propagation in solid media
- Solution to wave equation
- What is a wave?
- Evanescent waves
- Boundary conditions determine shape of a wave

Thank you for your attention!

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