Math review

1 Logarithms

Introduction and definition:

Logarithms are another way of thinking about exponents. The base a logarithm of a number N is defined as the exponent x that we need to raise the base in order to get the number, i.e.

$$\log_b(x) = y \Leftrightarrow b^y = x \tag{1}$$

Examples:

$$\begin{aligned} \log_3 81 = y \Leftrightarrow 3^y = 81 \rightarrow 3^y = 3^4 \rightarrow y = 4 \\ \log_2 128 = y \Leftrightarrow 2^y = 128 \rightarrow 2^y = 2^7 \rightarrow y = 7 \\ \log_3 \sqrt{243} = y \Leftrightarrow 3^y = (243)^{(1/2)} \rightarrow 3^y = (3^5)^{(1/2)} \rightarrow 3^y = 3^{(5/2)} \rightarrow y = \frac{5}{2} \end{aligned} \tag{2}$$

Properties of logarithms:

- Product: the logarithm of the multiplication of x and y is the sum of logarithm of x and logarithm of y, i.e.

$$\log_b(x \cdot y) = \log_b(x) + \log_b(y) \tag{3}$$

- Quotient: the logarithm of the division of x and y is the difference of logarithm of x and logarithm of y. i.e.

$$\log_b(x/y) = \log_b(x) - \log_b(y) \tag{4}$$

- Power: the logarithm of x raised to the power of y is y times the logarithm of x:

$$\log_b(x^y) = y \cdot \log_b(x) \tag{5}$$

- Base switch: the base b logarithm of c is 1 divided by the base c logarithm of b:

$$\log_b(c) = 1/\log_c(b) \tag{6}$$

- Base change: the base b logarithm of x is base c logarithm of x divided by the base c logarithm of b.

$$\log_b(x) = \log_c(x)/\log_c(b) \tag{7}$$

- Derivative:

$$f(x) = \log_b(x) \rightarrow f'(x) = 1/(x \ln(b)) \tag{8}$$

- Integral:

$$\int \log_b(x)dx = x \cdot (\log_b(x) - (1/\ln(b)) + C \tag{9}$$

Some direct consequences of the previous properties are:

- The logarithm of 1 (regardless of its base) is zero, i.e. $\log_b 1 = 0$
- The logarithm of the base itself equals one, i.e. $\log_b b = 1$
- It is only possible to calculate logarithm of positive numbers, i.e. $\log_b(x) = \nexists$ for $x \leq 0$
- The logarithm of infinity reads: $\log_b(x) = \infty$ when $x \to \infty$

Logarithm as inverse function of exponential function:

The logarithmic function $y = \log_b(x)$ is the inverse function of the exponential function $x = b^y$; i.e. $x = \log^{-1}(y) = b^y$. So if one calculates the exponential function of the logarithm of x ($x \leq 0$): $f(f^{-1}(x)) = b^{\log_b(x)} = x$. Or, if one calculates the logarithm of the exponential function of x, $f^{-1}(f(x)) = \log_b(b^x) = x$. One can see graphically how the logarithm function and its inverse look like in Figure 2.

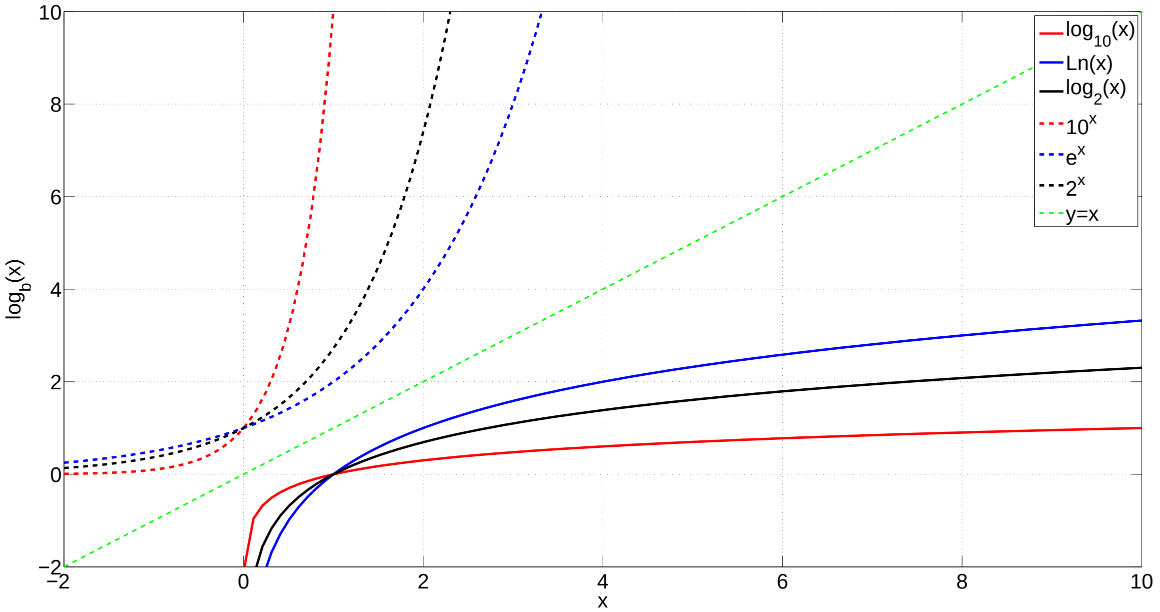

Figure 1: Different logarithm functions with its inverse.

Logarithm examples (and answers):

- **Example 1:** Solve $\log_2(x) + \log_2(x-3) = 2$

  $\log_2(x) + \log_2(x-3) = 2 \Rightarrow \log(x(x-3)) = 2 \Rightarrow x(x-3) = 2^2 \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow x_{1,2} = -1, 4$

  But since the logarithm is not defined for negative numbers, the answer is $x = 4$.

- **Example 2:** Find x for $\log_3(x+2) - \log_3(x) = 2$

  $\log_3((x+2)/x) = 2 \Rightarrow (x+2)/x = 3^2 \Rightarrow (x+2) = 9x \Rightarrow x = 0.25$

2 The decibel

Decibel (dB) is a logarithmic unit (i.e. dimensionless); it is a way to express a ratio or a gain. Decibel can be employed to express numerous quantities, e.g. the level of acoustic waves, electronic signals, velocity, acceleration, etc. By using a logarithmic scale, one can describe very big or very small numbers with shorter notation. Particularly in acoustics, this is very useful, since our inner ear can detect differences in pressure ranging from 20μPa to 200 Pa. The latter would imply a large "axis" when representing those quantities. However, if decibels are used, that scale becomes much more manipulable (i.e. easy to use and grasp). Note that in order to express a quantity in decibels, a reference quantity (often agreed upon by the research community) must be considered. In the case of acoustics and for the calculation of sound pressure level, for example, 20μPa is adopted, as it is the smallest sound pressure a healthy young inner ear can detect. All in all, the dB level can be viewed as relative gain of one level versus other level, or absolute logarithmic scale level for well known reference levels.

A ratio (in bels) is defined as the base 10 logarithm of the ratio of two quantities, namely P_1 and the reference in question considered, P_0, i.e.

$$\text{Ratio}_B = \log_{10} = (P_1/P_0). \tag{10}$$

Decibel is one tenth of a Bel, so 1 B= 10 dB.

Power ratio: The power ratio in decibels (dB) is 10 times base 10 logarithm of the ratio of P_1 and P_0,

$$\text{Ratio}_{dB} = \log_{10} = (P_1/P_0) \tag{11}$$

Amplitude ratio: The ratio of quantities like voltage, electric current, acceleration, sound pressure level... are calculated as ratio of squares. The amplitude ratio in decibels (dB) is 20 times base 10 logarithm of the ratio of V_1 and V_0, i.e.

$$\text{Ratio}_{dB} = 10\log_{10} = (P_1^2/P_0^2) = 20\log_{10} = (V_1/V_0) \tag{12}$$

3 Complex oscillations

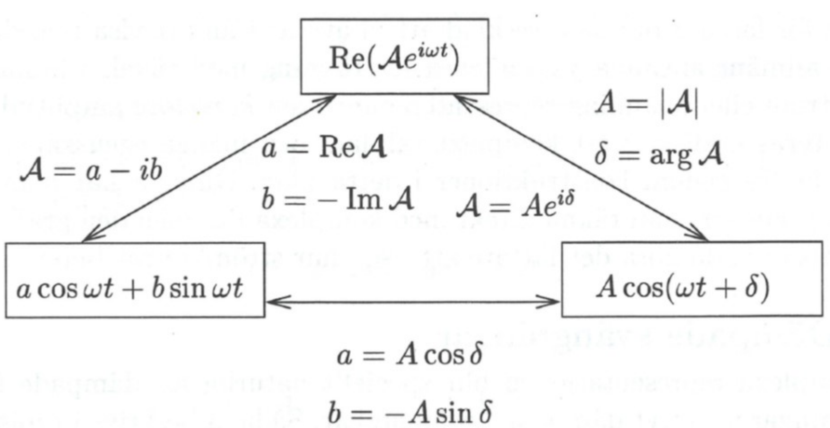

Figure 2: Summary of the relationships between different representations of complex oscillations.

As we will further see in the course, an oscillation can be represented in many different ways, e.g. in the time or frequency domain, and therefore there exist different mathematical ways to describe them, e.g. trigonometrically, exponentially, etc. A good graphical summary to keep in mind througout the course between different mathematical relationships is shown in Figure **??** [1].

4 Exercises

1. Calculate x, by applying the logarithm definition:

   - $\log_2 64 = x$
   - $\log_2 \sqrt{8} = x$
   - $\log_{(1/2)} 4 = x$
   - $\log_x 125 = 3$
   - $\log_3 x = 3$

   Solutions: $x = 6; x = 3/2; x = -2; x = 5; x = 27$.

2. Solve, without calculator and by applying the logarithm definition:

   - $\log_2 64 + \log_2(1/4) - \log_3 9 - \log_2 \sqrt{2}$
   - $\log_2 \left(\frac{1}{32}\right) + \log_3 \left(\frac{1}{27}\right) - \log_2 1$

   Solutions: $x = 3/2; x = -8$.

3. Calculate and express the result in logarithmic form:

   - $\log_3 5 + \log_3 6$
   - $\log_2 30 - \log_2 15$
   - $\log_4 x^5$

   Solutions: $\log_3 30; 1; 5 \log_4 x$.

4. Calculate without the calculator:

   - $\log 1000 - \log 0.001 + \log\left(\frac{1}{1000}\right)$
   - $\log 7 + \log\left(\frac{1}{7}\right)$

   Solutions: $3; 0$.

[1] [1] Sven Spanne: Komplex Analys. Lunds Universitet.