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RECORDING

# Ljud i byggnad och samhälle (VTAF01) – Waves in solids

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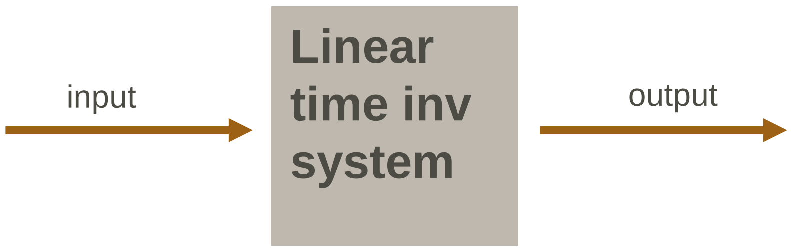
# Recap from previous lecture F2

- Recap...

# Linear system

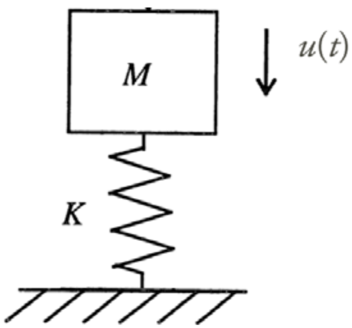
- What effect does an input signal has on an ouput signal?
  - What effect does a force on a body has on its velocity?
- A way to answer it using theory of linear time-invariant systems.



# Linear system

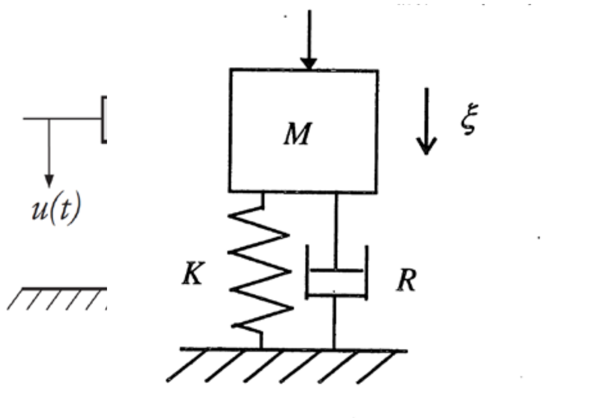
- Linear time-invariant systems
  - Mathematically: relation input/output described by linear differential equations.
  - Characteristics:
    - » Coefficients independent of time.
    - » Superposition principle.
      - $a(t) \rightarrow c(t); b(t) \rightarrow d(t) \Rightarrow a(t) + b(t) \rightarrow c(t) + d(t)$
    - » Homogeneity principle: $\alpha a(t) \rightarrow \alpha$b(t)
    - » Frequency conserving:
      - a(t) comprises frequencies f1 and f2; b(t) comprises f1 and f2.
  - Linear oscillations shall be considered

# Equations of Motions of a mass-spring system




- Forces in the system:
  - Newton's law: $M\ddot{x}$
  - Hooke's law: $-Kx$
- Forces shall balance each other:
  - $M\ddot{x} = -Kx$
  - $M\ddot{x} + Kx = 0$
- We got Equations of Motions (EoM) of the system!
- $x(t) = ae^{\lambda t}$
- $\lambda_1 = \mathrm{i}\sqrt{\frac{K}{M}} = \mathrm{i}\omega_0$ ; $\lambda_2 = -\mathrm{i}\sqrt{\frac{K}{M}} = -\mathrm{i}\omega_0$.
- $x(t) = ae^{\mathrm{i}\omega_0 t} + be^{-\mathrm{i}\omega_0 t} = \mathrm{A}\sin(\omega_0 t) + \mathrm{B}\cos(\omega_0 t)$.

# Free vibrations with damping



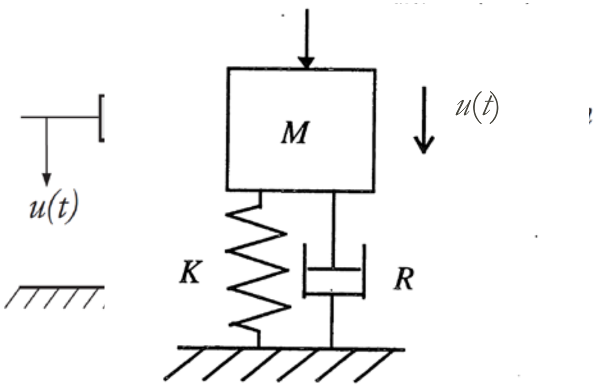

- $\lambda_1 = -\mathrm{R} + \mathrm{i}\omega_R = \mathrm{i}\omega_0$ ; $\lambda_2 = -\mathrm{R} - \mathrm{i}\omega_R$ ; $\omega_R = \sqrt{\omega_0^2 - R^2}$.
- $x(t) = ae^{-\mathrm{R}t+\mathrm{i}\omega_R t} + be^{-\mathrm{R}t-\mathrm{i}\omega_R t} =$

  $= [\mathrm{A}\sin(\omega_R t) + \mathrm{B}\cos(\omega_R t)]\, e^{-\mathrm{R}t}$

# Forced motion – Damped SDOF

- Mass-spring-damper system (e.g. a floor)

$F(t) = F_{driv} \cdot \cos(\omega t)$




$$M\ddot{u}(t) + R\dot{u}(t) + Ku(t) = F(t)$$

Inertial force — Damping force — Elastic force — Applied force

  - $u(t)$ obtained by solving the EoM together with the initial conditions
    - » Solution = Homogeneous + Particular

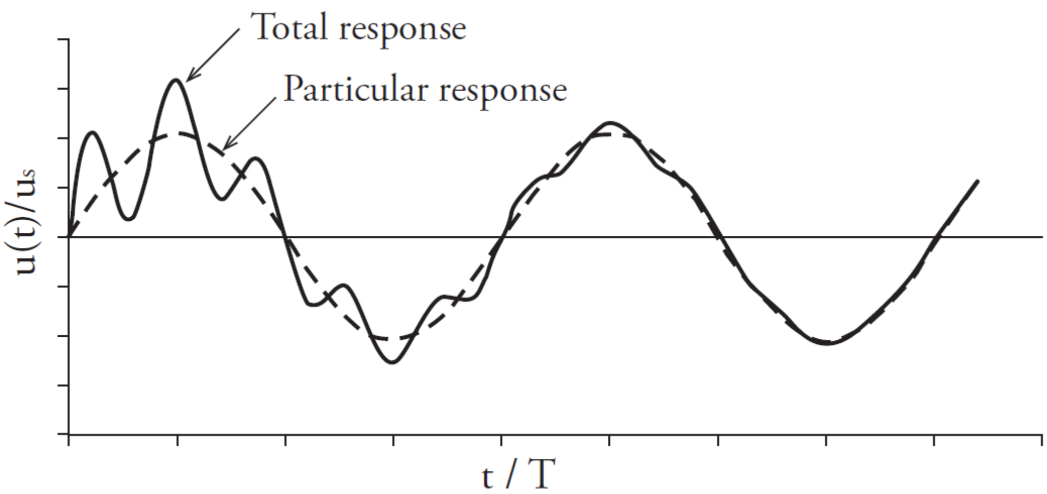
$$u(t) = u_p(t) + u_h(t)$$

$u_p(t)$: $F(t) = F_{driv} \cdot \cos(\omega t)$ — $u_h(t)$: $F(t) = 0$

NOTE: Damping is the energy dissipation of a vibrating system

# Damped SDOF – Total solution

- Total solution = homogeneous + particular
  - The homogeneous solution vanishes with increasing time. After some time: $u(t) \approx u_p(t)$




*Total response of a damped system subjected to a harmonic force,*

$$u_{total}(t) = u_p(t) + u_h(t) = \left[ \frac{\left(\frac{F_{driv}}{K}\right) \cdot \cos(\omega t - \alpha)}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2 \cdot \left(\frac{\eta}{2}\right) \cdot \left(\frac{\omega}{\omega_0}\right)\right)^2}} \right] + e^{-\frac{\eta}{2}\omega_0 t}\left(B_1 \sin(\omega_d t) + B_2 \cos(\omega_d t)\right)$$



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# SDOF – Complex representation (Freq. domain)

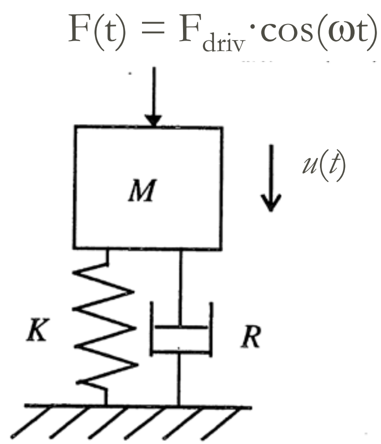
- Euler's formula: $e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$

- Then: $F(t) = F_{driv}\cos(\omega t) = \mathrm{Re}\left[F_{driv}e^{i\omega t}\right]$

  $u(t) = u_0\cos(\omega t - \varphi) = \mathrm{Re}\left[u e^{i\varphi}e^{i\omega t}\right] = \mathrm{Re}\left[\tilde{u}(\omega)e^{i\omega t}\right]$

- Differenciating: $\dot{u}(t) = \mathrm{Re}\left[i\omega \cdot \tilde{u}(\omega)e^{i\omega t}\right]$

  $\ddot{u}(t) = \mathrm{Re}\left[-\omega^2 \cdot \tilde{u}(\omega)e^{i\omega t}\right]$




- Substituting in the EOM: $M\ddot{u}(t) + R\dot{u}(t) + Ku(t) = F_{driv}\cos(\omega t)$

NOTE: Differential equation became second order equation with time-harmonic ansats!

$$\tilde{u}(\omega) = \frac{F_{driv}}{(K - M\omega^2) + Ri\omega}$$

NOTE: This is the **particular** solution in complex form for a damped SDOF system. In Acoustics, most of the times, we are interested in the particular solution, which is the one not vanishing as time goes by.

If the system is excited with $\omega_0^2 = K/M$
→ Resonance (dominated by damping)

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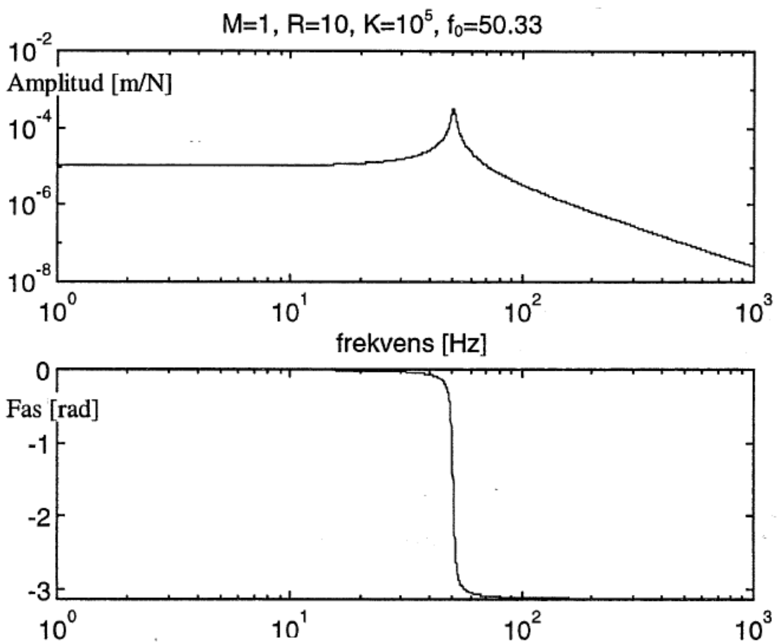
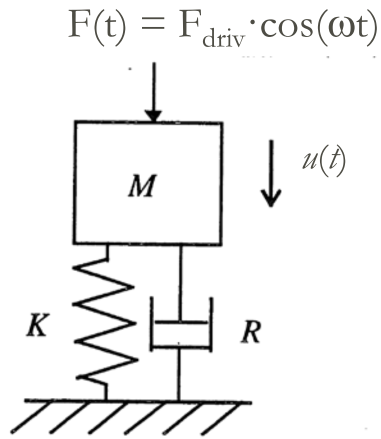
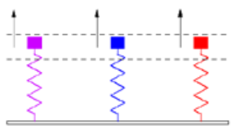
# SDOF – Frequency response function

- Output / Input $$\frac{\tilde{u}(\omega)}{F_{driv}} = \frac{1}{(K - M\omega^2) + Ri\omega}$$
- Kvoten är ett komplext tal och heter överföringsfunktion



# SDOF – Frequency response functions (FRF)

- In general, FRF = transfer function, i.e.:
  - Contains system information
  - Independent of outer conditions
  - **Frequency domain relationship between input and output of a linear time-invariant system**

$$H_{ij}(\omega) = \frac{\tilde{s}_i(\omega)}{\tilde{s}_j(\omega)} = \frac{\text{output}}{\text{input}}$$


- Different FRFs can be obtained depending on the measured quantity

| Measured quantity | FRF | |
|---|---|---|
| Acceleration (a) | Accelerance = $N_{dyn}(\omega)$ = a/F | Dynamic Mass = $M_{dyn}(\omega)$ = F/a |
| Velocity (v) | Mobility/admitance = $Y(\omega)$ = v/F | Impedance = $Z(\omega)$ = F/v |
| Displacement (u) | Receptance/compliance = $C_{dyn}(\omega)$= u/F | Dynamic stiffness = $K_{dyn}(\omega)$ = F/u |

$$C_{dyn}(\omega) = \frac{\tilde{u}(\omega)}{F_{driv}(\omega)} = \frac{1}{(K - M\omega^2) + Ri\omega}$$

$$K_{dyn}(\omega) = C_{dyn}(\omega)^{-1} = -M\omega^2 + Ri\omega + K$$



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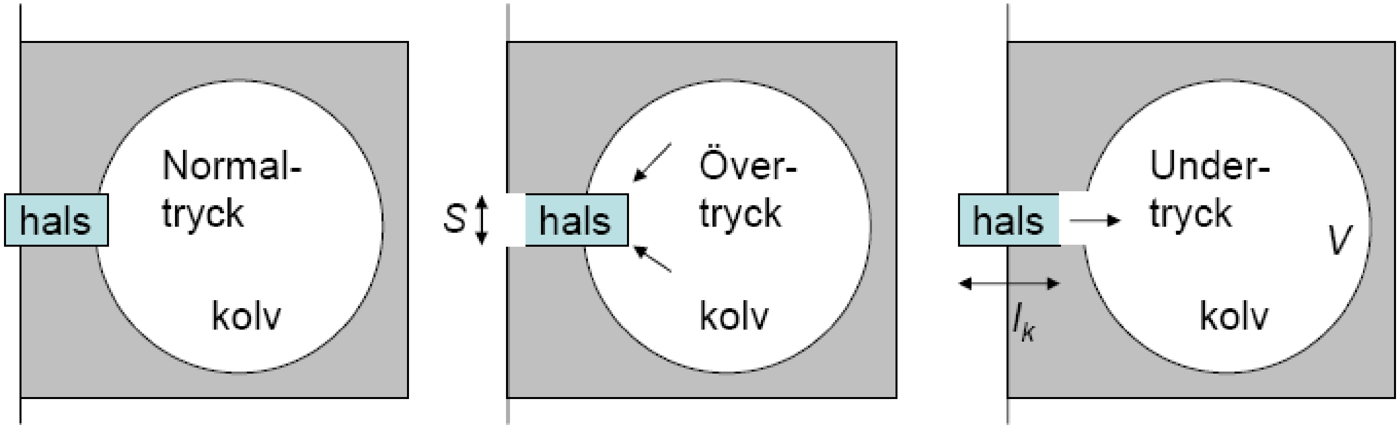
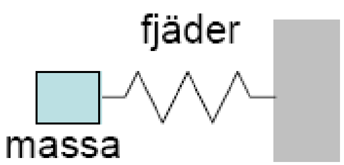
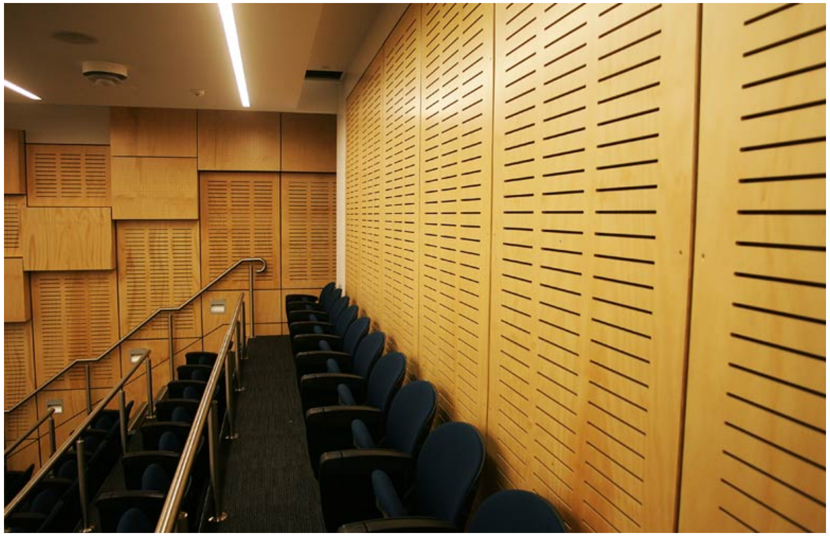
# Helmholtz resonator




$$f_r = \frac{c}{2\pi}\sqrt{\frac{S}{l_k V}}$$

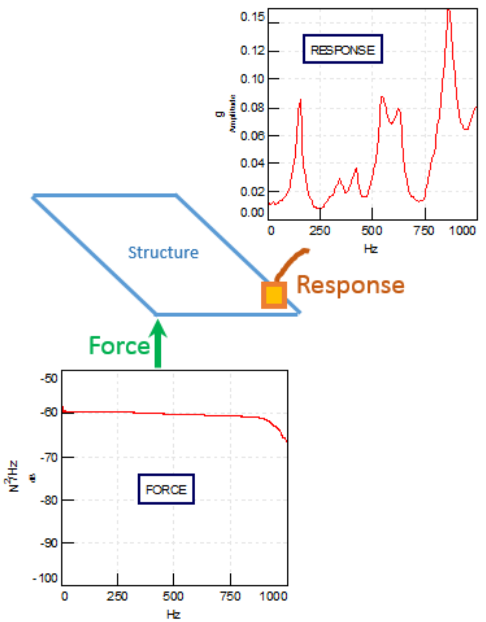
# Helmholtz resonator (IV)

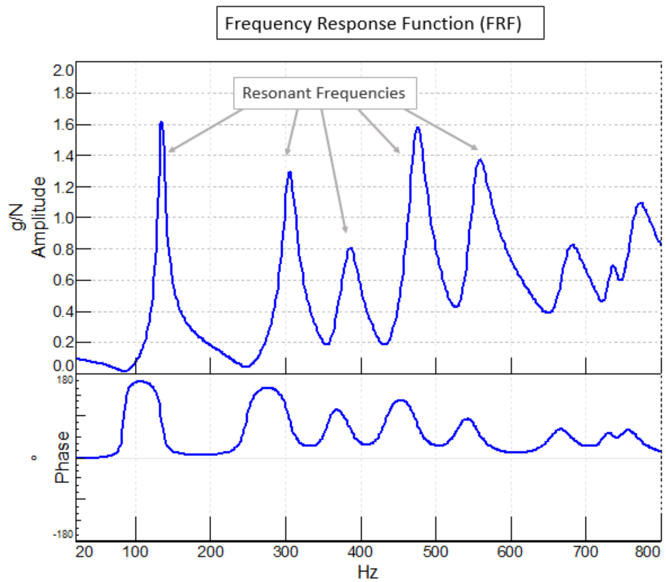


# FRF of a complex system

- What happens if we evaluate FRF of a complex (linear) system such as a plate?
  - We gain useful info on it!

$$H_{ij}(\omega) = \frac{\tilde{s}_i(\omega)}{\tilde{s}_j(\omega)} = \frac{\text{output}}{\text{input}}$$


Source: https://community.sw.siemens.com/s/article/what-is-a-frequency-response-function-frf



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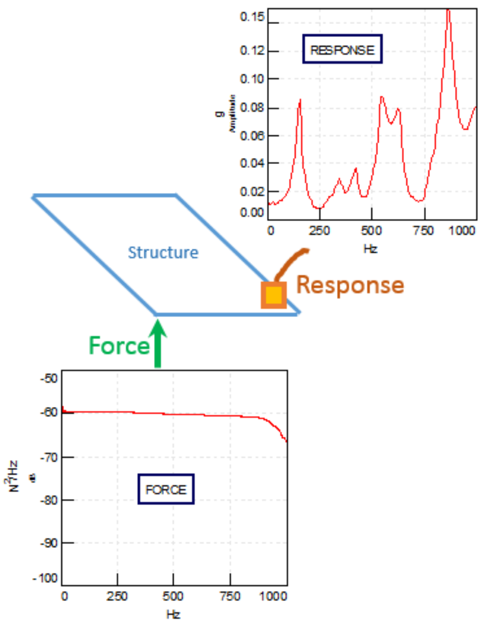
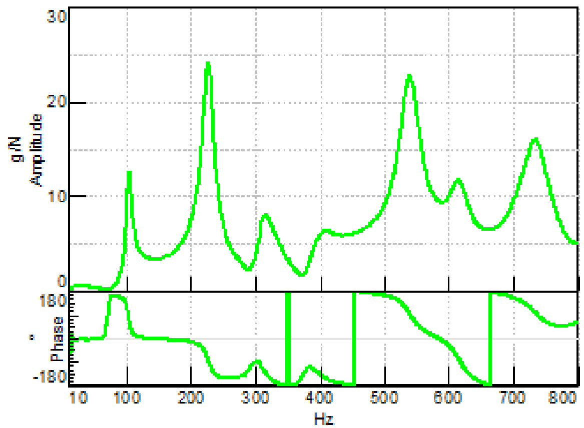
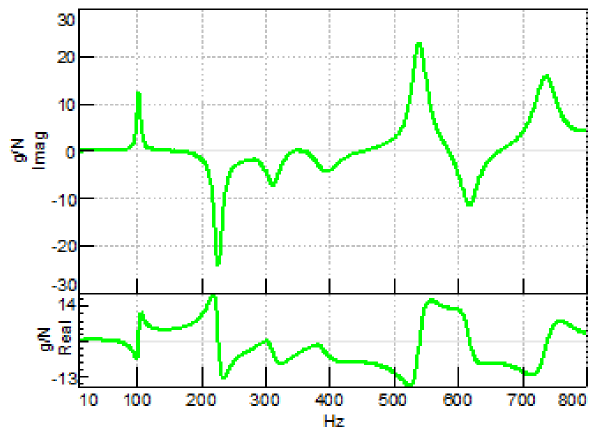
# FRF of a complex system

- FRFs are complex
  - Amplitude/Phase
  - Real / imaginary part

$$H_{ij}(\omega) = \frac{\tilde{s}_i(\omega)}{\tilde{s}_j(\omega)} = \frac{\text{output}}{\text{input}}$$

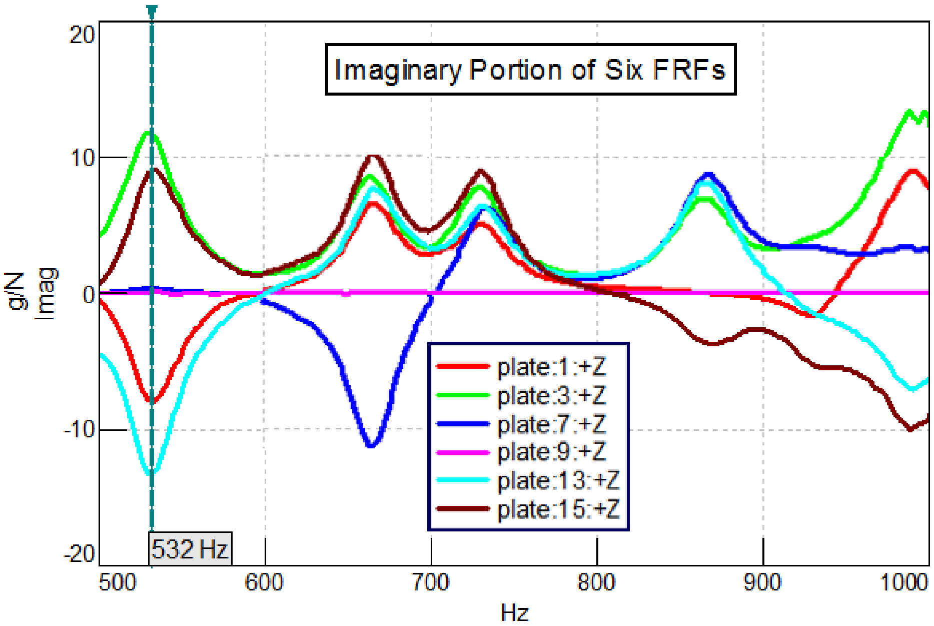



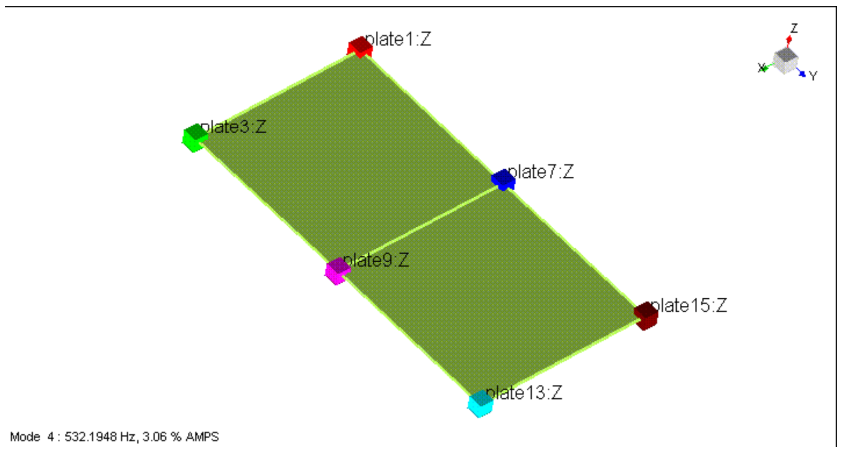

$$\lambda_1 = -\mathrm{R} + i\omega_R = i\omega_0 \; ; \; \lambda_2 = -\mathrm{R} - i\omega_R \; ; \; \omega_R = \sqrt{\omega_0^2 - R^2}.$$

Source: https://community.sw.siemens.com/s/article/what-is-a-frequency-response-function-frf

# FRF of a complex system

- Real and imaginary parts – the imaginary part has interesting information

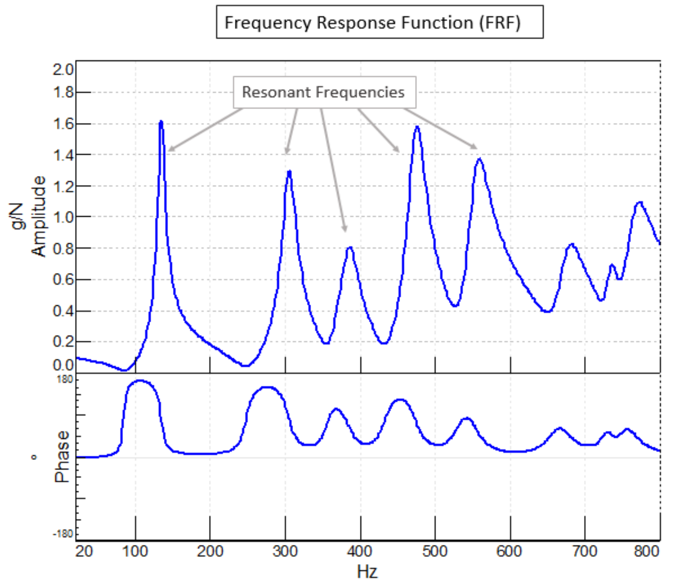



Source: https://community.sw.siemens.com/s/article/what-is-a-frequency-response-function-frf

# FRF of a complex system

- Each peak is showing a natural frequency
    - Each peak is a mass-spring-damper SDOF system?!

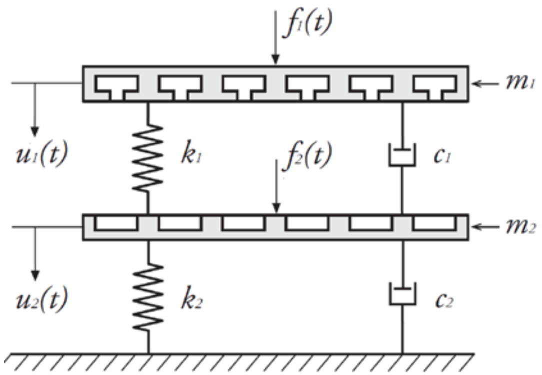



Source: https://community.sw.siemens.com/s/article/what-is-a-frequency-response-function-frf

# MDOF – Multi-degree-of-freedom systems

- In reality, more DOFs are needed to define a system → MDOFs
  - Continuous systems → often approximated by MDOFs

- Multi-degree-of-freedom system (Mass-spring-damper)
  - Solution process: similar as in SDOFs (particular+homogeneous)




$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t)$$

$$\mathbf{M} \in \mathbb{R}^{n\times n}$$
$$\mathbf{K} \in \mathbb{R}^{n\times n}$$
$$\mathbf{C} \in \mathbb{R}^{n\times n}$$
$$\mathbf{f}(t) \in \mathbb{R}^{n\times 1}$$
$$\mathbf{u}(t) \in \mathbb{R}^{n\times 1}$$

  - *"The undamped modes form an orthogonal basis, i.e. they uncouple the system, allowing the solution to be expressed as a sum of the eigenmodes of the free-vibration SDOF system"*

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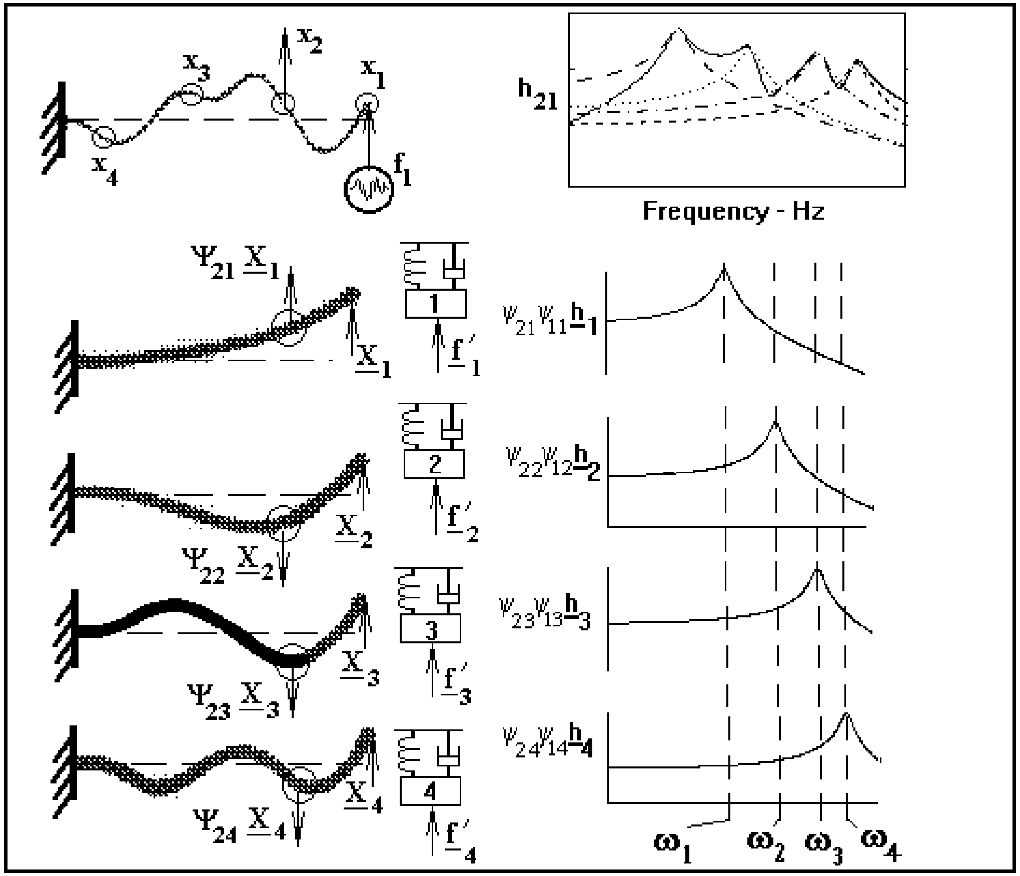
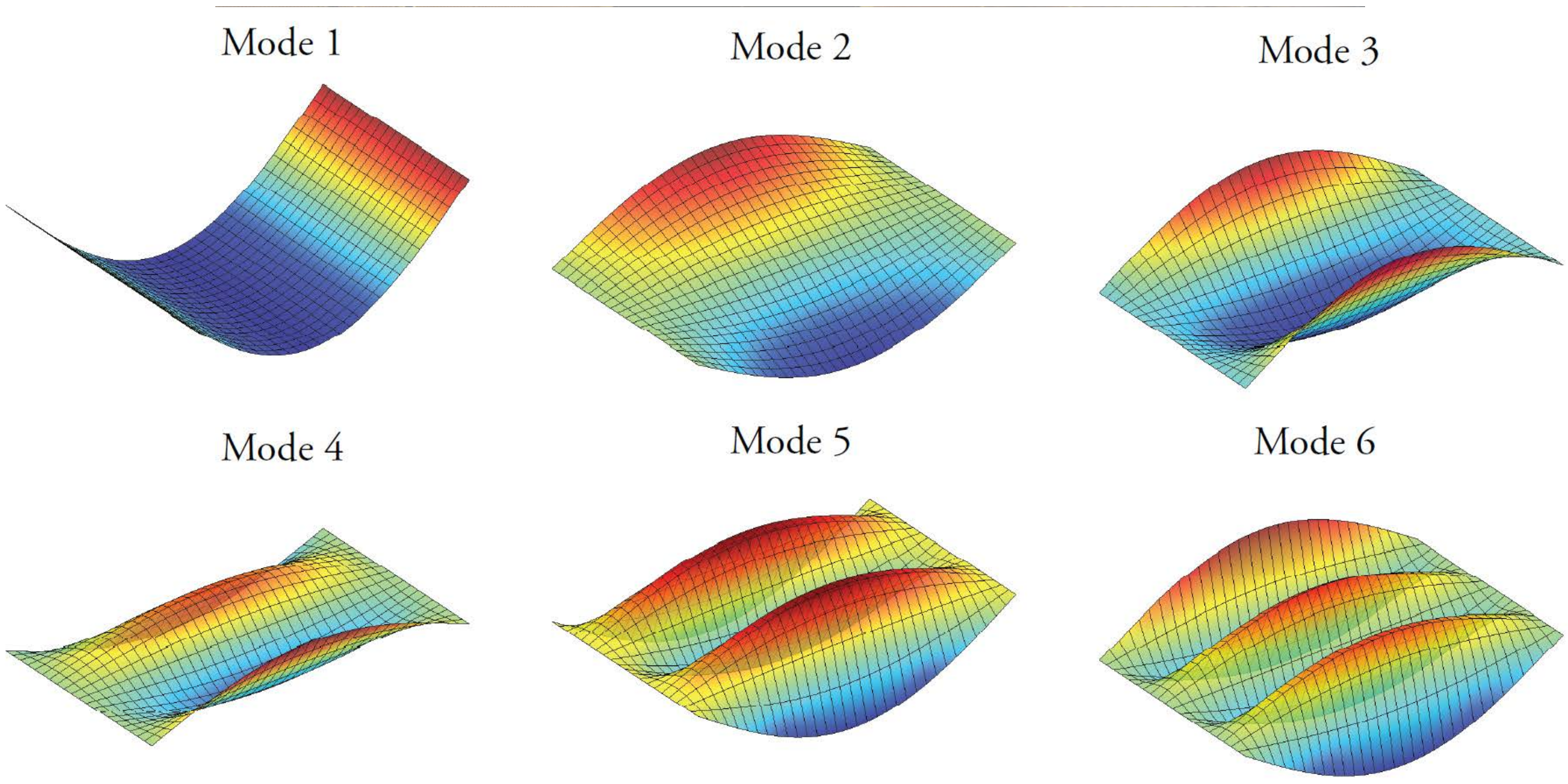
# MDOF – Note on modal superposition




*Source:* http://signalysis.com

# Mode shapes – Example floor




NOTE: In floor vibrations, modes are superimposed on one another to give the overall response of the system. Fortunately it is generally sufficient to consider only the first 3 or 4 modes, since the higher modes are quickly extinguished by damping.

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# Resonance & Eigenmodes

Examples:

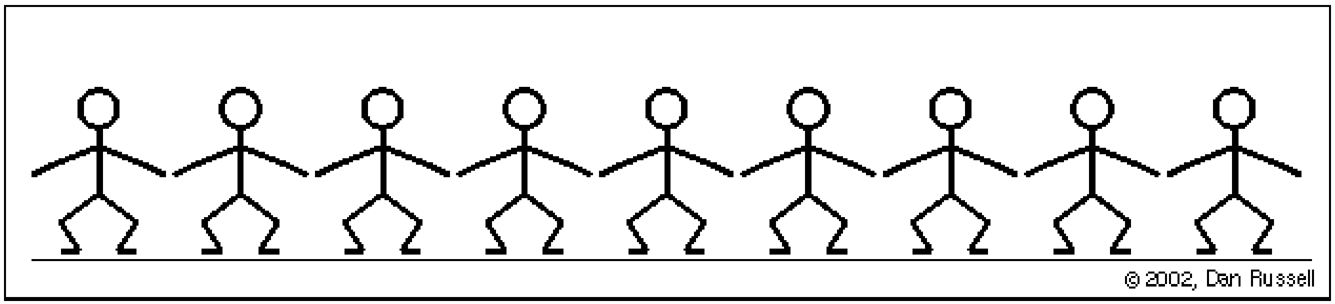
- Earthquake design
- Bridges (Tacoma & Spain)
- Modes of vibration: Plate

# Recap from previous lecture F2

- End of recap...

# Learning outcomes

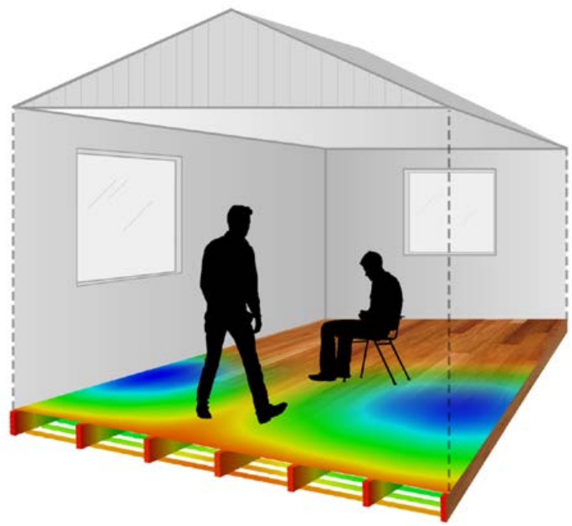
- Wave propagation in solid media
- Wave equation solution




**Wave is a disturbance that travels in space!**
**People jumps up and sits down. None is carried away with the wave.**

# Introduction

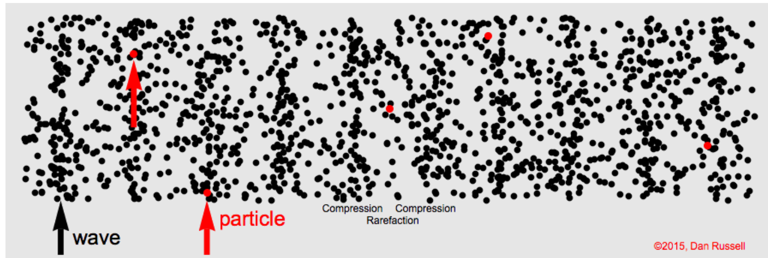
- A very broad definition…
  - Acoustics: *what can be heard*…
  - Vibrations: *what can be felt*…
- Coupled "problem"
  - Hard to draw a line between both domains
- Nuisance to building users
  - Comprise both noise and vibrations

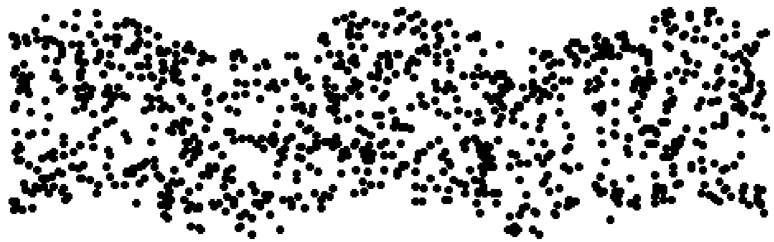


*Source: J. Negreira (2016)*

# Types of waves – classification

- Depending on propagation media
  - Mechanical waves (solids and fluids)
  - Electromagnetical waves (vacuum)
- Propagation direction
  - 1D, 2D and 3D
- Based on periodicity
  - Periodic and non-periodic
- Based on particles' movement in relation with propagation direction:
  - Longitudinal waves (solids and fluids)
  - Transverse waves (solids)
- More?



<u>NOTE:</u> waves do not transport mass, just energy

# Types of waves in solid media

- Longitudinal waves
- Shear waves
- Torsional waves
- Bending waves
- Rayleigh waves
- Lamb waves
- …

# Derivation of longitudinal wave equations (I)

- General approach to derive equations of motion:

1. Newton's law – dynamic equilibrium
2. Constitutive relations – forces, stresses and strains
    - Relations between two physical quantities in a material
    a. Force – stress
    b. Stress – strain
3. Strain – displacement relation (definition)

# Derivation of longitudinal wave equations (II)

- Stress: *Dragspänning eller normalspänning definieras som den negativa spänning som uppstår i en enaxligt belastad stång utsatt för en dragkraft. Krafterna normeras med planets yta, så att dessa spänningar har enheten för tryck. Vanligtvis betecknas dragspänning med den grekiska bokstaven sigma.*
- Strain: *Töjning (elongation) är ett enhetslöst, geometrioberoende mått på deformationsgraden och betecknas ε (epsilon). Töjning kan anta både positiva och negativa värden beroende på om objektet utsatts för drag- eller tryckspänning.*
- Constitutive relations: *Förhållanden mellan två fysiska kvantiteter i ett visst material.*

Source: Wikipedia.se



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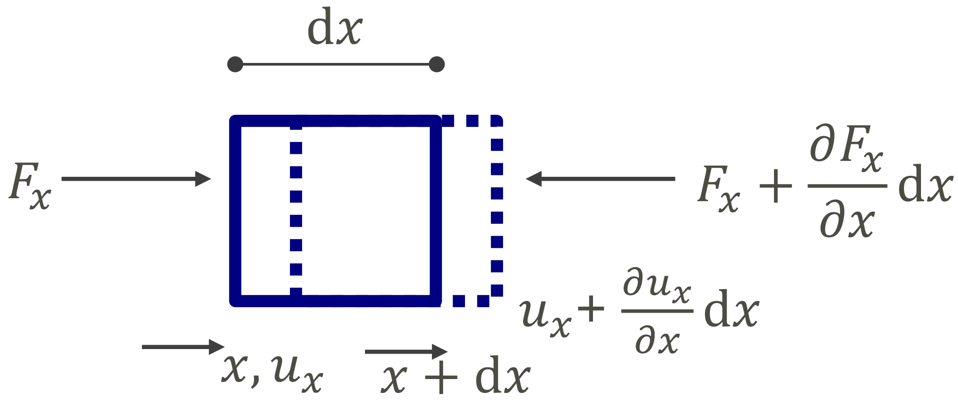
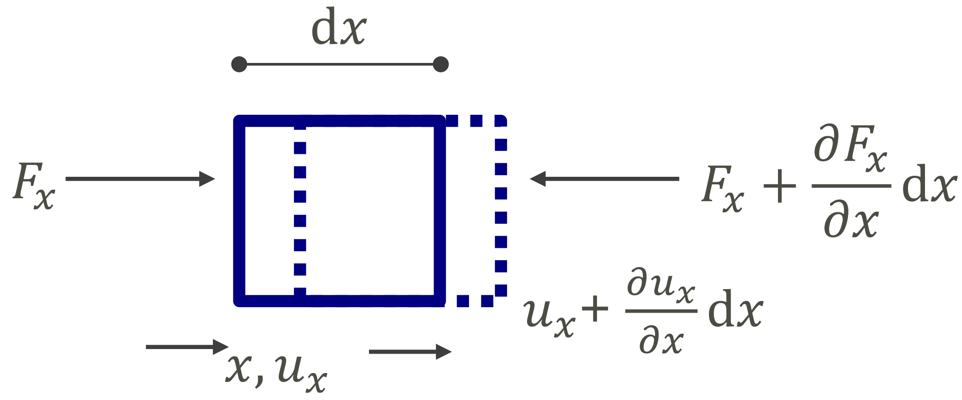
# Derivation of longitudinal wave equations



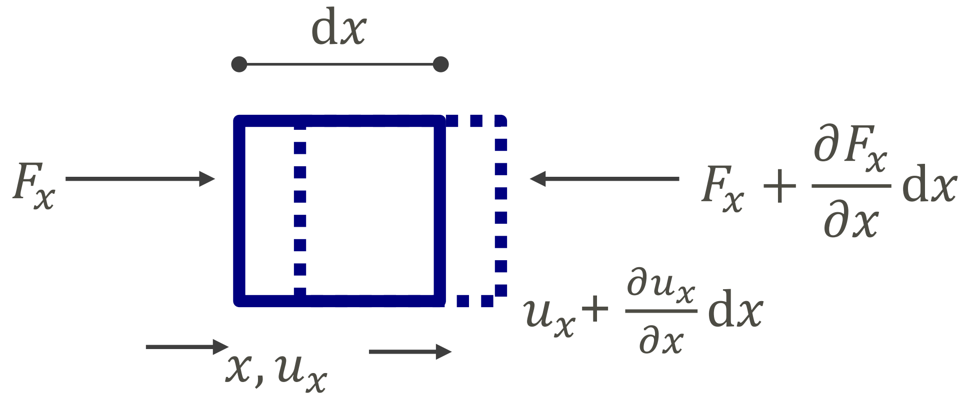

- We take a small piece of a bar with mass density $\rho$, Young's modulus $E$, dx long and with section S.

# Derivation of longitudinal wave equations



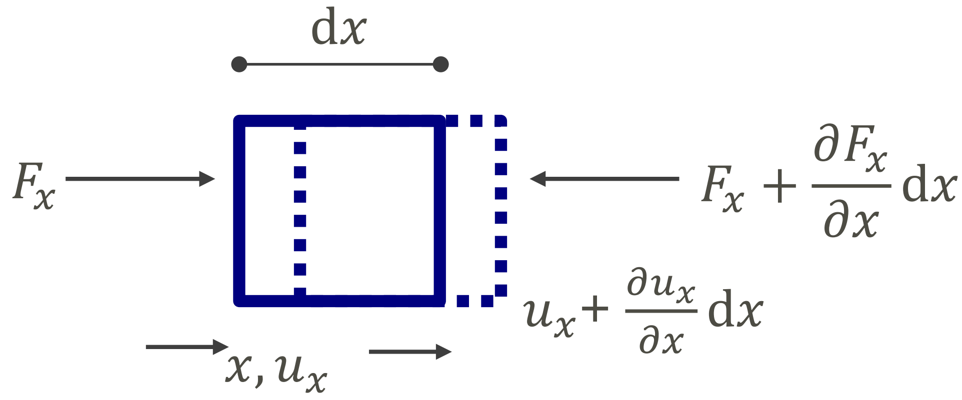

- Balance of forces
  - $\left(F_x + \frac{\partial F_x}{\partial x}\mathrm{d}x\right) - F_x = ma = -\rho S \mathrm{d}x \frac{\partial^2 u_x}{\partial t^2}$
- Constitutive relation (relation between two physical quantities in a material)
  - $\sigma_x = E\varepsilon_x$
    - $\left[\frac{N}{m^2} = \frac{kg\frac{m}{s^2}}{m^2} = \frac{kg}{ms^2}\right] = \left[Pa = \frac{N}{m^2}\right] [-]$

# Derivation of longitudinal wave equations




- Force-stress
  - $F_x = -\sigma_x S$
- Strain-displacement
  - $\varepsilon_x = \frac{\partial u_x}{\partial x}$
- *After some easy but tedious mathematical steps to rearrange equations…*

# Derivation of longitudinal wave equations

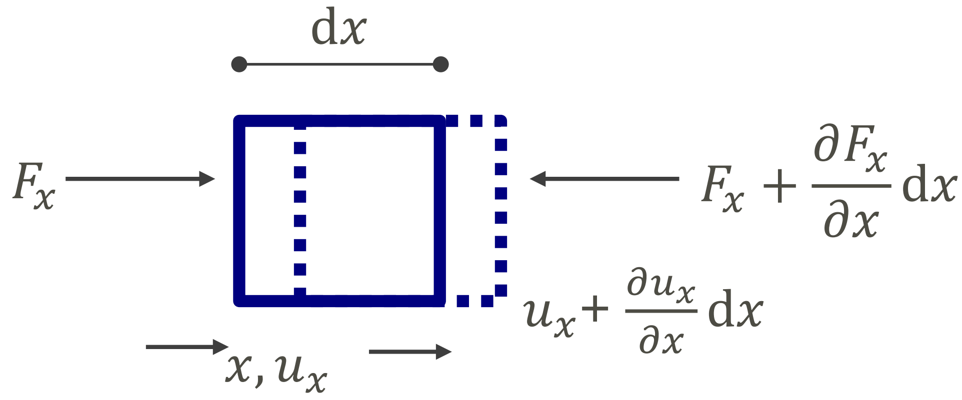



- Equations of motions for a bar (ideal longitudinal waves)

  - $$\frac{\partial^2 u_x}{\partial x^2} - \frac{\rho}{E}\frac{\partial^2 u_x}{\partial t^2} = 0; c = \sqrt{\frac{E}{\rho}}$$

  - Second order partial differential equations
    - Solved with 2 initial conditions (in time) and two boundary conditions (in space)

# Derivation of longitudinal wave equations




- Equations of motions for a bar (ideal longitudinal waves)
  - $\frac{\partial^2 u_x}{\partial x^2} - \frac{\rho}{E}\frac{\partial^2 u_x}{\partial t^2} = 0$
  - $\frac{\partial^2 v_x}{\partial x^2} - \frac{\rho}{E}\frac{\partial^2 v_x}{\partial t^2} = 0$
  - $\frac{\partial^2 F_x}{\partial x^2} - \frac{\rho}{E}\frac{\partial^2 F_x}{\partial t^2} = 0$



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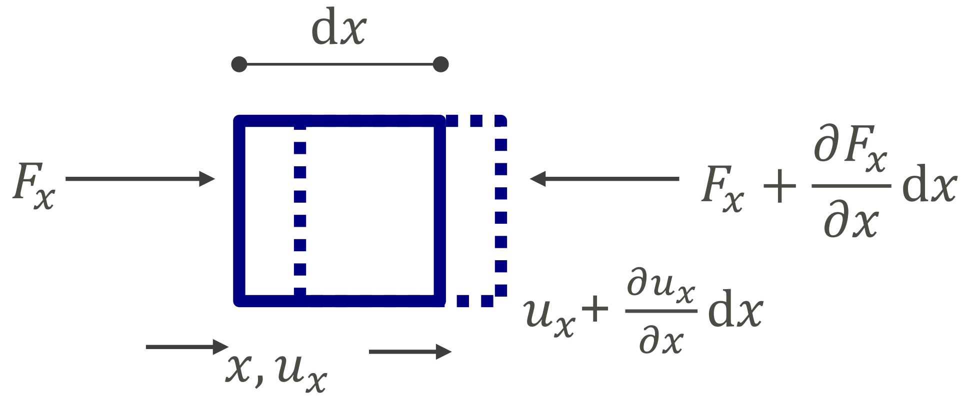
# Derivation of longitudinal wave equations



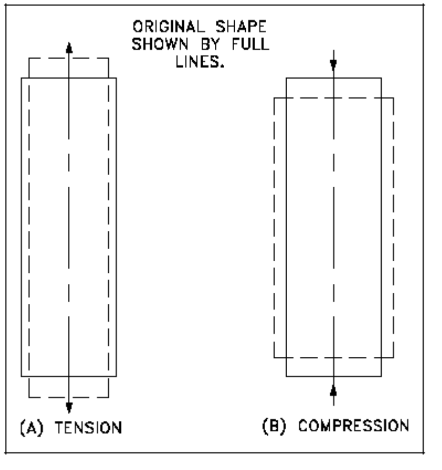

- Equations of motions for a bar (longitudinal waves)

  - $$\frac{\partial^2 u_x}{\partial x^2} - \frac{\rho}{E(1-\nu^2)}\frac{\partial^2 u_x}{\partial t^2} = 0$$

  - Where $\nu$ is Poisson's ratio

$$\nu = -\frac{\text{Strain in direction of load}}{\text{Strain at right angle to load}}$$

$$\nu = -\frac{\varepsilon_{lateral}}{\varepsilon_{axial}}$$




https://www.engineersedge.com/material_science/poissons_ratio_definition_equation_13159.htm

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# General form of a wave equation

One dimension: $\frac{\partial^2 u_x}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2} = 0$

Three dimensions: $c^2\nabla^2 u - \ddot{u}=0$

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f \qquad \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Laplacian

**https://en.wikipedia.org/wiki/Laplace_operator**

- It took some physics reasoning and some math but now we have an expression that we can use with most wave types that are relevant in acoustics and vibrations!
  - Not however with the most important structural waves in acoustics, which is a bit special!

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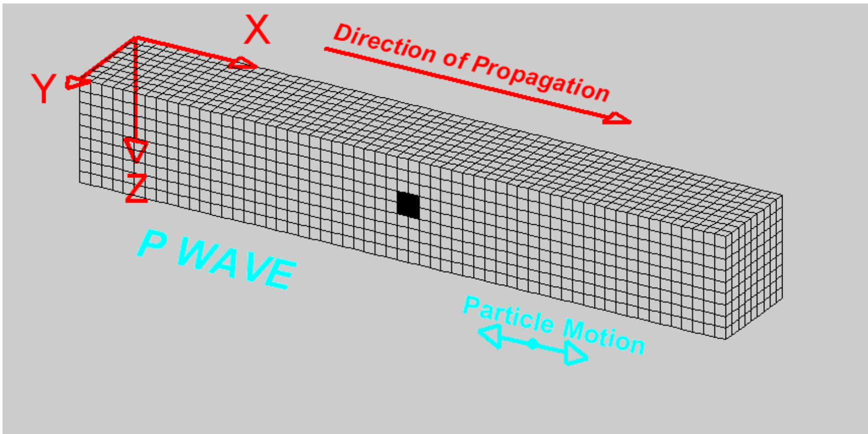
# Waves in solid media

- Now we can go through some kinds of waves in solid media!

# Longitudinal waves

- P waves (primary waves in seismology)




http://www.geo.mtu.edu/UPSeis/waves.html

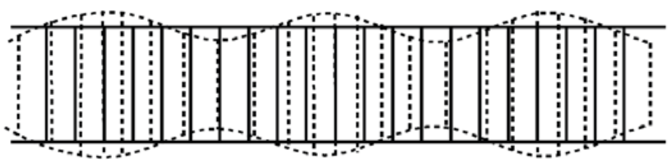
# Longitudinal waves

- Longitudinal waves (∞ medium ≈ beams)
  - Quasi-longitudunal waves (finite ≈ plates)

$$\frac{\partial^2 u_x}{\partial x^2} - \frac{\rho}{E'}\frac{\partial^2 u_x}{\partial t^2} = 0$$

$$c_L = \sqrt{\frac{E}{\rho}}$$

$$c_{qL} = \sqrt{\frac{E'}{\rho}} = \sqrt{\frac{E}{\rho(1-\upsilon^2)}}$$




- *Longitudinell våg, en våg där punkterna i vågmediet svänger i vågens utbredningsriktning. Härvid komprimeras mediet, och den återställande kraften ges av tryck. Ett exempel är en ljudvåg eller en vanlig fjäder.*
  - *Motsatsen är en transversell våg där punkterna i vågmediet svänger vinkelrätt mot utbredningsriktningen. Exempel: stränginstrument, vattnet i en damm och elektromagnetisk strålning.*

Source: Wikipedia.se

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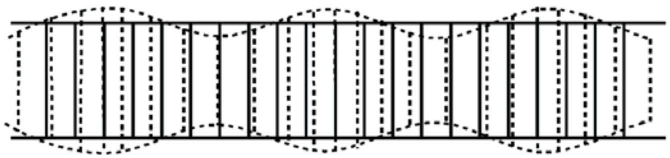
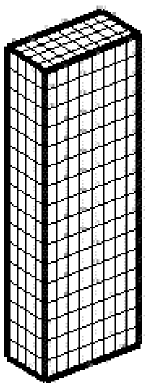
# Longitudinal waves

- Longitudinal waves (∞ medium ≈ beams)
  - Quasi-longitudunal waves (finite ≈ plates)

Plate: E, G, ρ, υ, h

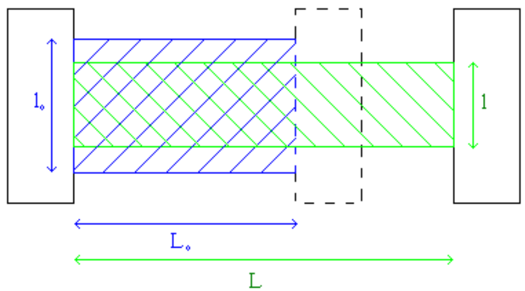




- – Poissons konstant, Poissons tal eller tvärkontraktionstalet är en materialkonstant som anger hur ett material reagerar på tryck- och dragkrafter. När ett material (blått) töjs ut i en riktning dras det ihop i andra riktningar (grönt).

$$c_L = \sqrt{\frac{E}{\rho}}$$

$$c_{qL} = \sqrt{\frac{E'}{\rho}} = \sqrt{\frac{E}{\rho(1-\upsilon^2)}}$$




Source: Wikipedia.se

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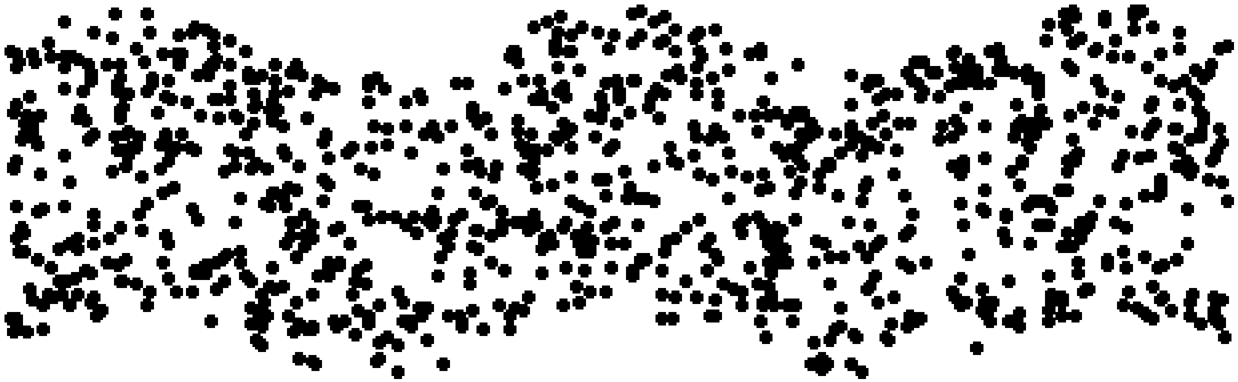
# Shear waves

- Shear waves / transverse waves



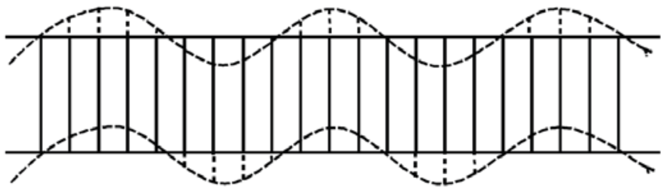
# Shear waves

- Shear waves

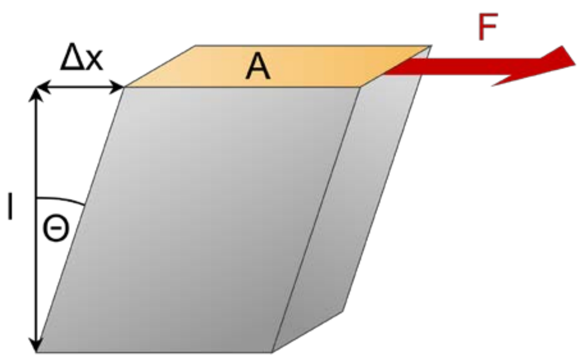
$$\frac{\partial^2 u_y}{\partial x^2} - \frac{\rho}{G}\frac{\partial^2 u_y}{\partial t^2} = 0$$

$$c_{sh} = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{E}{2(1+\upsilon)\rho}}$$

Plate: E, G, ρ, υ, h



*Skjuvning, eller skjuvtöjning, är en deformation utan volymändring. Den definieras som vinkeländringen skapad av deformationen.*




Source: Wikipedia.se

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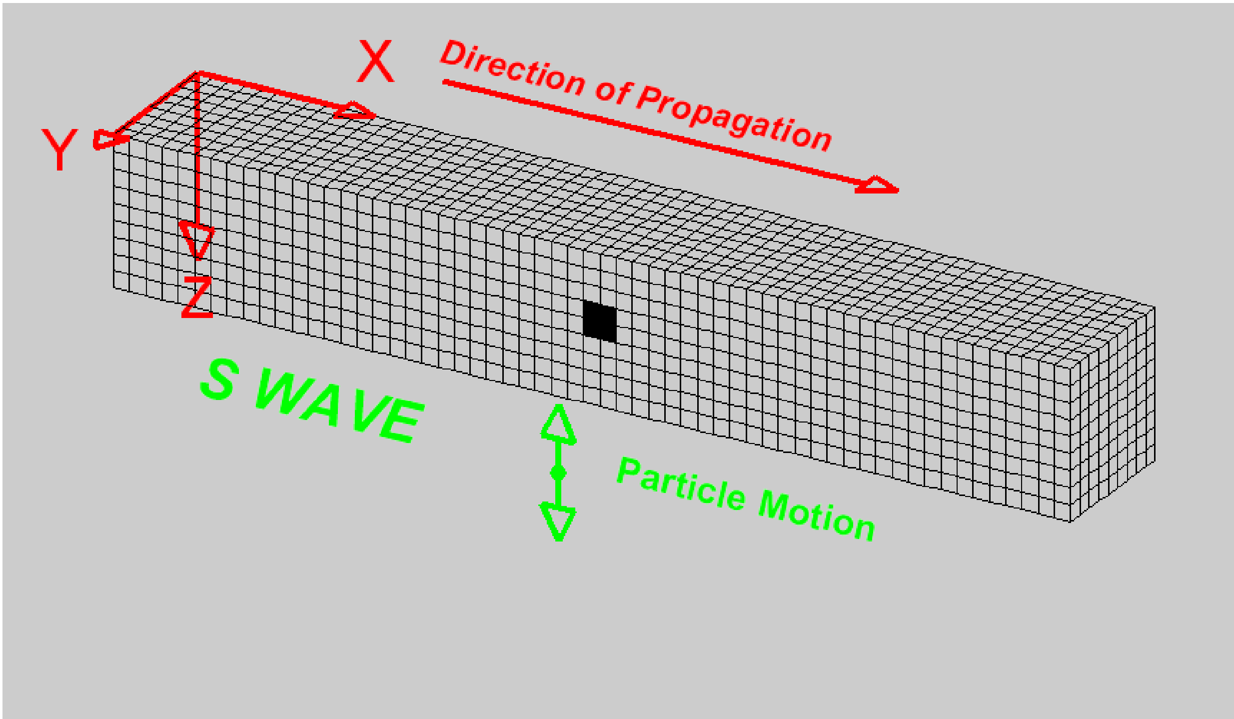
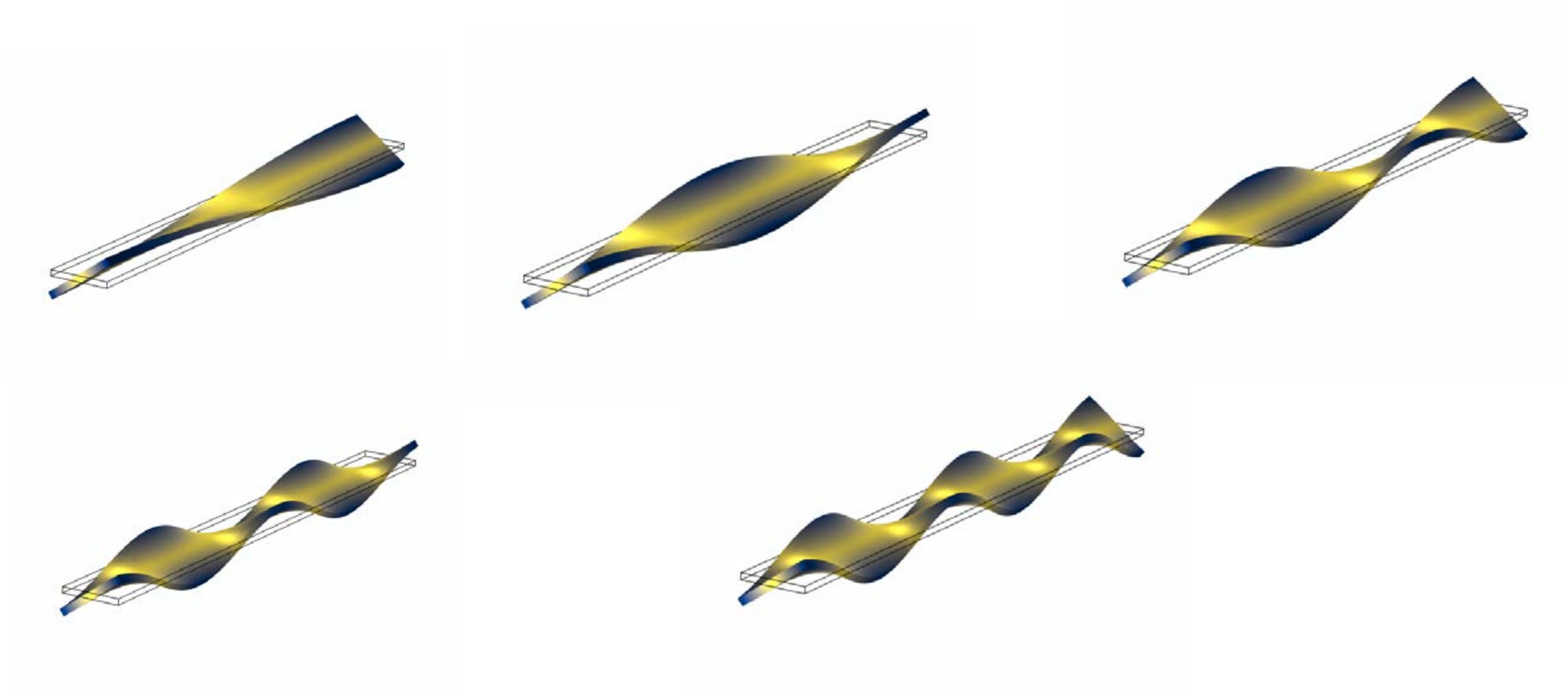
# Shear waves

- S waves



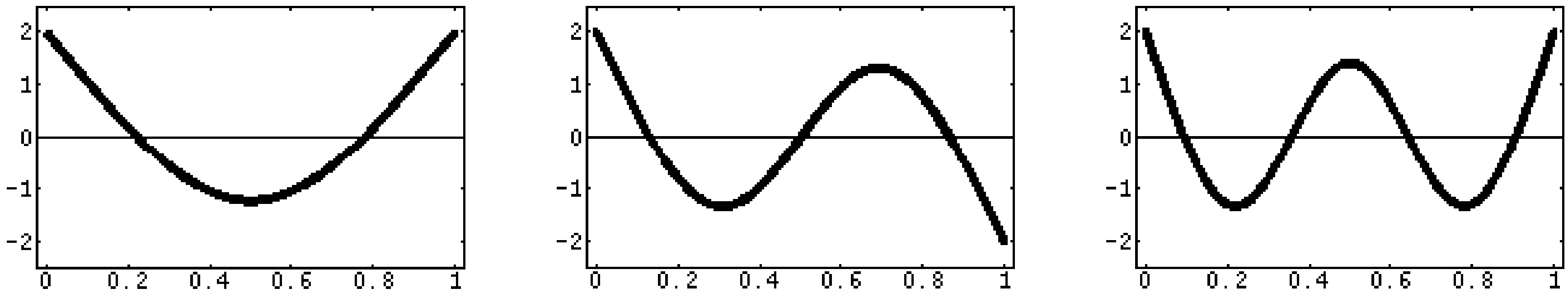
# Torsional waves

- Torsional waves $$\frac{\partial^2\theta}{\partial x^2} - \frac{1}{c_T^2}\frac{\partial^2\theta}{\partial t^2} = 0 \quad \text{with} \quad c_T = \sqrt{\frac{GK}{\rho I_p}}$$



# Bending waves

- Bending waves (free-free) – Böjvågor på svenska



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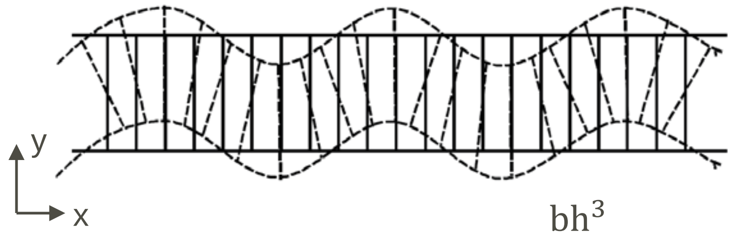
# Bending waves

- Bending (or flexural) waves (dispersive)

$$B\frac{\partial^4 u_y}{\partial x^4} + m\frac{\partial^2 u_y}{\partial t^2} = 0 \qquad c_{B(\omega)} = \sqrt{\omega}\sqrt[4]{\frac{B}{m}}$$

Plate: E, G, ρ, υ, h




$$m = \rho h \qquad B_{beam} = E\frac{bh^3}{12}$$

$$B_{plate} = \frac{Eh^3}{12(1-\upsilon^2)}$$

- Planar section remain plane
- Towards the middle line perpendicular cross-section remains perpendicular to the middle line after deformation (that is, shear deformation is neglected).
- Different form of wave equation!
- Happen when a transverse load is applied to a structure (beam, plate).
  - Examples?!
- Perhaps the most important structural wave in acoustics.
  - Next lecture F4 we will talk again about bending waves and explain why this is true. Spoiler: bending waves radiate sound well.

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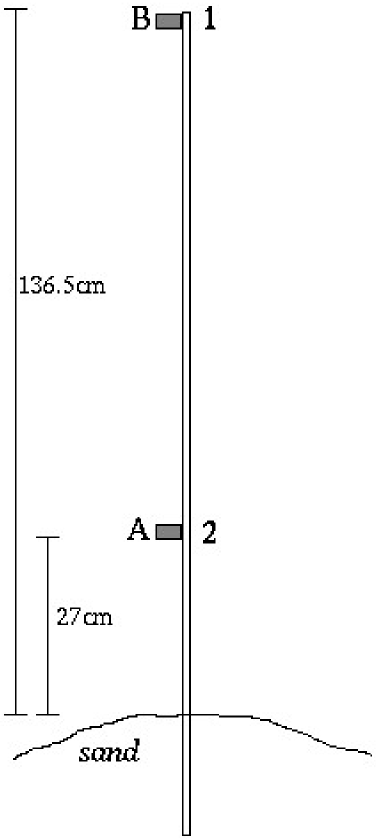
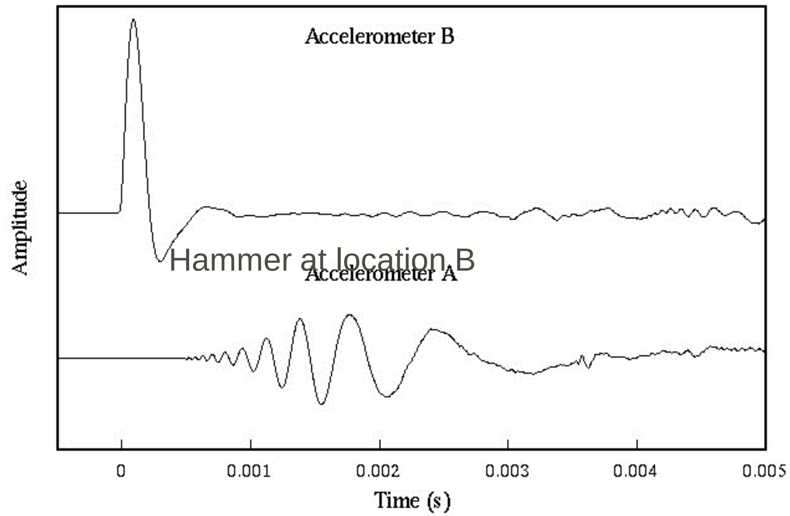
# Bending waves

$$B\frac{\partial^4 u_y}{\partial x^4} + m\frac{\partial^2 u_y}{\partial t^2} = 0 \qquad c_{B(\omega)} = \sqrt{\omega}\sqrt[4]{\frac{B}{m}}$$




- Force pulse is very clean at location B
- Pulse disperses by the time it reaches location A --- higher frequency waves travel faster and arrive first --- lower frequency waves travel slower and arrive later

**https://www.acs.psu.edu/drussell/Demos/Dispersion/Flexural.html**



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# Recap - Types of waves in solid media

- Longitudinal waves (∞ medium ≈ beams)
  - Quasi-longitudunal waves (finite ≈ plates)

$$\frac{\partial^2 u_x}{\partial x^2} - \frac{\rho}{E'}\frac{\partial^2 u_x}{\partial t^2} = 0$$

$$c_L = \sqrt{\frac{E}{\rho}}$$

$$c_{qL} = \sqrt{\frac{E'}{\rho}} = \sqrt{\frac{E}{\rho(1-\upsilon^2)}}$$

- Shear waves

$$\frac{\partial^2 u_y}{\partial x^2} - \frac{\rho}{G}\frac{\partial^2 u_y}{\partial t^2} = 0$$

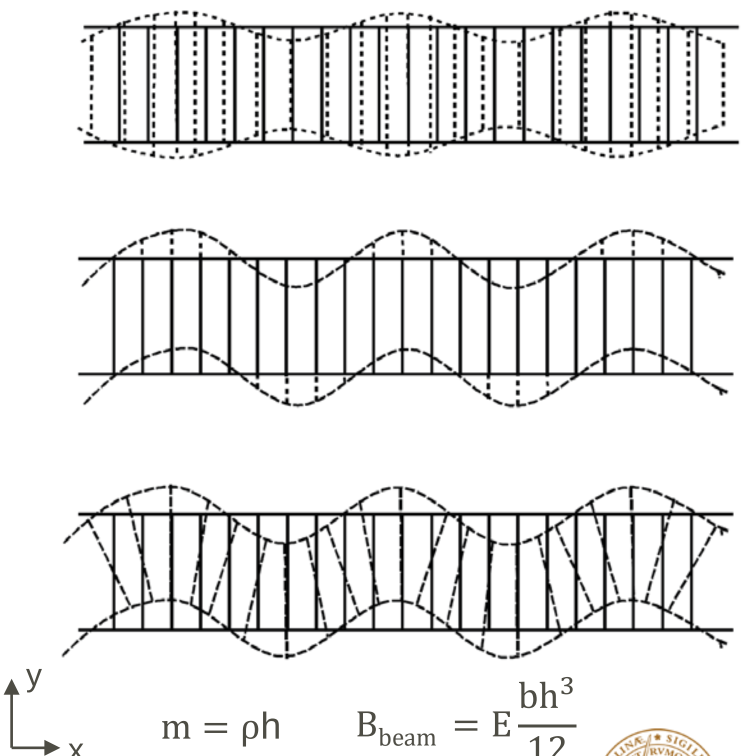
$$c_{sh} = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{E}{2(1+\upsilon)\rho}}$$

- Bending waves (dispersive)

$$B\frac{\partial^4 u_y}{\partial x^4} + m\frac{\partial^2 u_y}{\partial t^2} = 0$$

$$c_{B(\omega)} = \sqrt{\omega}\sqrt[4]{\frac{B}{m}}$$

Plate: E, G, ρ, υ, h




$$m = \rho h \qquad B_{beam} = E\frac{bh^3}{12}$$

$$B_{plate} = \frac{Eh^3}{12(1-\upsilon^2)}$$

<u>NOTE:</u> torsional waves (beams and columns) are not addressed here



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# Solution to the wave equation

One dimension: $\frac{\partial^2 u_x}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2} = 0$

- What is the simplest shape of a sound wave?
  - An harmonic shape of sinusoidal shape
    - » $u = A\sin a(x - ct)$; $u = A\cos a(x - ct)$
  - It follows that $u = Ae^{\pm ia(x-ct)}$ is also a solution.
  - As $u = A\ln a(x - ct)$; or $u = A\sqrt{a(x - ct)}$, which are not oscillatory.
- It turns out that any function of the form $u = f(x - ct)$ is solution.
- No assumption is made on f – except on its argument.
  - What does that imply?

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# Solution to the wave equation

One dimension: $\frac{\partial^2 u_x}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2} = 0$

- A wave is translated, unchanged in shape, along x (space)
- A given point on a wave is translated unchanged with speed *c*.
- This operation defines propagation!

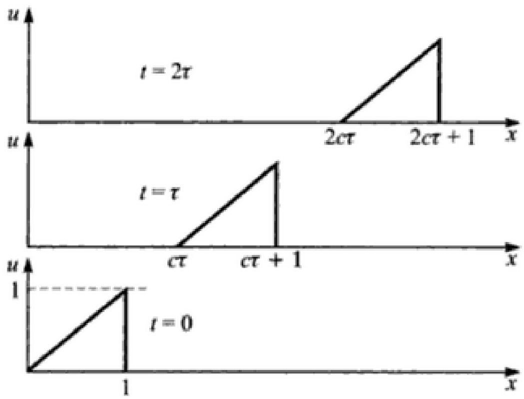
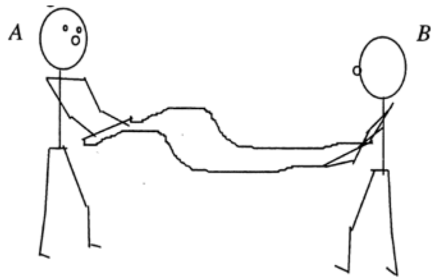



**Figure 1.1** Sketches showing the waveform in space when the solution is a section of a ramp function.

Blackstock, Fundamentals of physical acoustics

# Solution to the wave equation




- Pg 25 of the compendium

Vi kan tänka oss två personer som skakar en matta, vi kallar dem A och B. Vi tänker oss vidare att B håller sin kant stilla medan A gör en plötslig rörelse uppåt vid sin kant. Resten av mattan vill nu följa med i denna rörelse, med början med de punkter på mattan som är närmast A. Rörelsen fortsätter sedan att sprida sig med en konstant hastighet tills den når B. Om A fortsätter att skaka sin ände upp och ned, och provar olika takt i skakandet, olika frekvens, så kommer de finna att om man skakar snabbt så blir ”pulsen”, eller våglängden, kort och om man skakar långsamt så blir våglängden lång. Men oavsett vilken frekvens de skakar med så kommer spridningshastigheten att vara densamma. Med andra ord, händelsen att ”röra sig uppåt” sprider sig längs mattan med en viss hastighet som vi kan kalla $c$, vågutbredningshastighet. Det känns naturligt att $c$ beror på mattans vikt och hur hårt A och B drar i mattan, hur stor spänningen är. Är massan stor, tung matta, transporteras vågen långsamt. Är spänningen stor går vågen snabbt. Den initiala förskjutningen kommer att repeteras vid en punkt belägen en sträcka $x$ från A, och detta sker efter $x/c$ sekunder, det vill säga den tid det tar för vågen att utbreda sig sträckan $x$.

Det är viktigt att inse att ingen massa transporteras av vågen, vad som transporteras är endast möjligheten till rörelse. Massan i mattan rör sig endast upp och sedan ner igen.

I exemplet med mattan ovan var förskjutningen uppåt medan vågutbredningen går mellan A och B,

# Solution to the wave equation

One dimension: $\frac{\partial^2 u_x}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u_x}{\partial t^2} = 0$

- It turns out that any function of the form $u = f(x - ct)$ is solution.
- Prove it!
- Lead:

$$u(x,t) = u_+(t - x/c) \Rightarrow$$

$$\frac{\partial u}{\partial x} = -\frac{1}{c}u_+'(t - x/c) \quad ; \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}u_+''(t - x/c)$$

$$\frac{\partial u}{\partial t} = u_+'(t - x/c) \quad ; \quad \frac{\partial^2 u}{\partial t^2} = u_+''(t - x/c)$$

# What is a wave then?

- A disturbance or deviation from a pre-existing condition. Its motion constitutes a transfer of information from one point in space to another.
- Time plays a key role – static displacement of a rubber band is a disturbance but not a wave.
  - Wave travels at finite speed (hitting a perfectly rigid rod making the rod moving as a unit is no wave, just rigid body motion)
  - The rod is elastic, the impulse travels from one end to the other.
- All mechanical waves travel in a material medium (unlike e.g. electromagnetic waves)
- Many waves satisfy $c^2\nabla^2 u - \ddot{u}=0$ – but not all!

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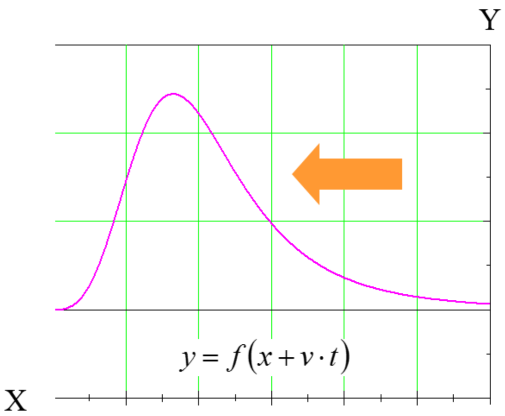
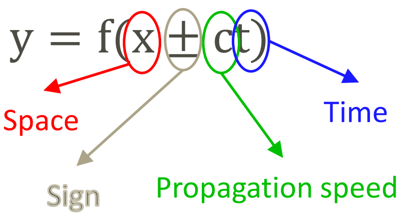
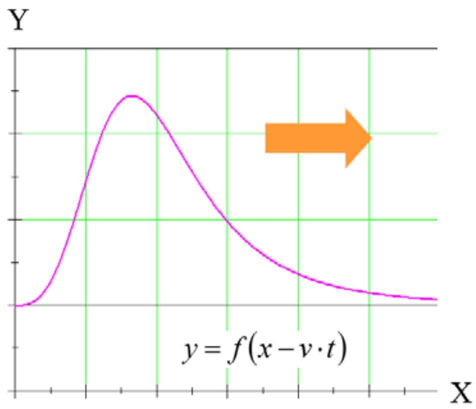
# Wave equation solution

- Most general solution is forward and backward travelling wave.
    - d'Alambert's solution




$$y = f\left(x \pm \frac{\omega}{k} t\right) = f\left(\frac{kx \pm \omega t}{k}\right) = f(kx \pm \omega t)$$

- Alternative forms:

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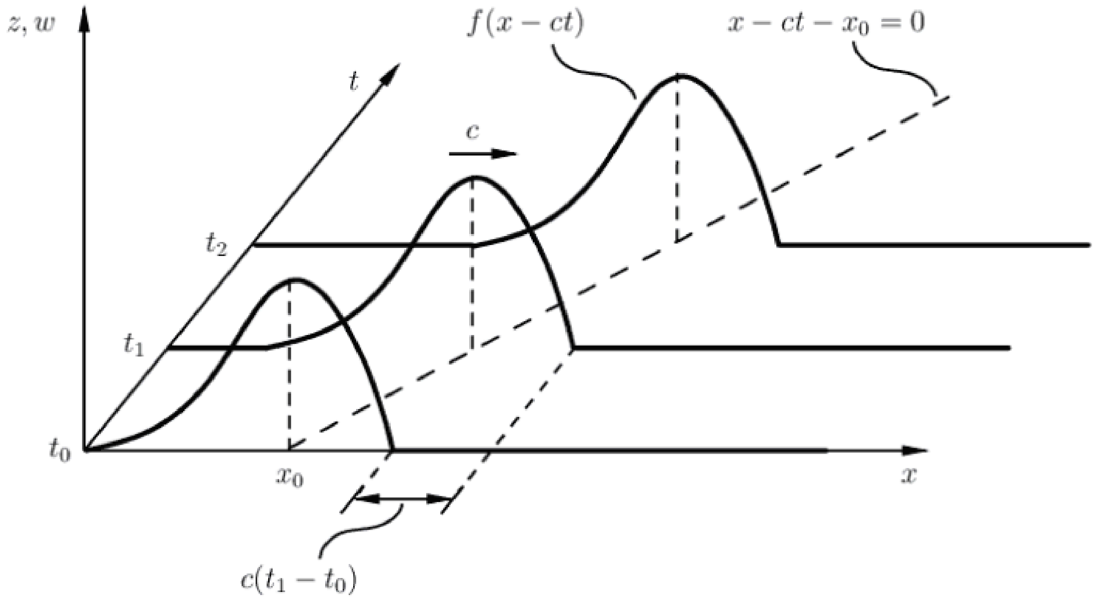
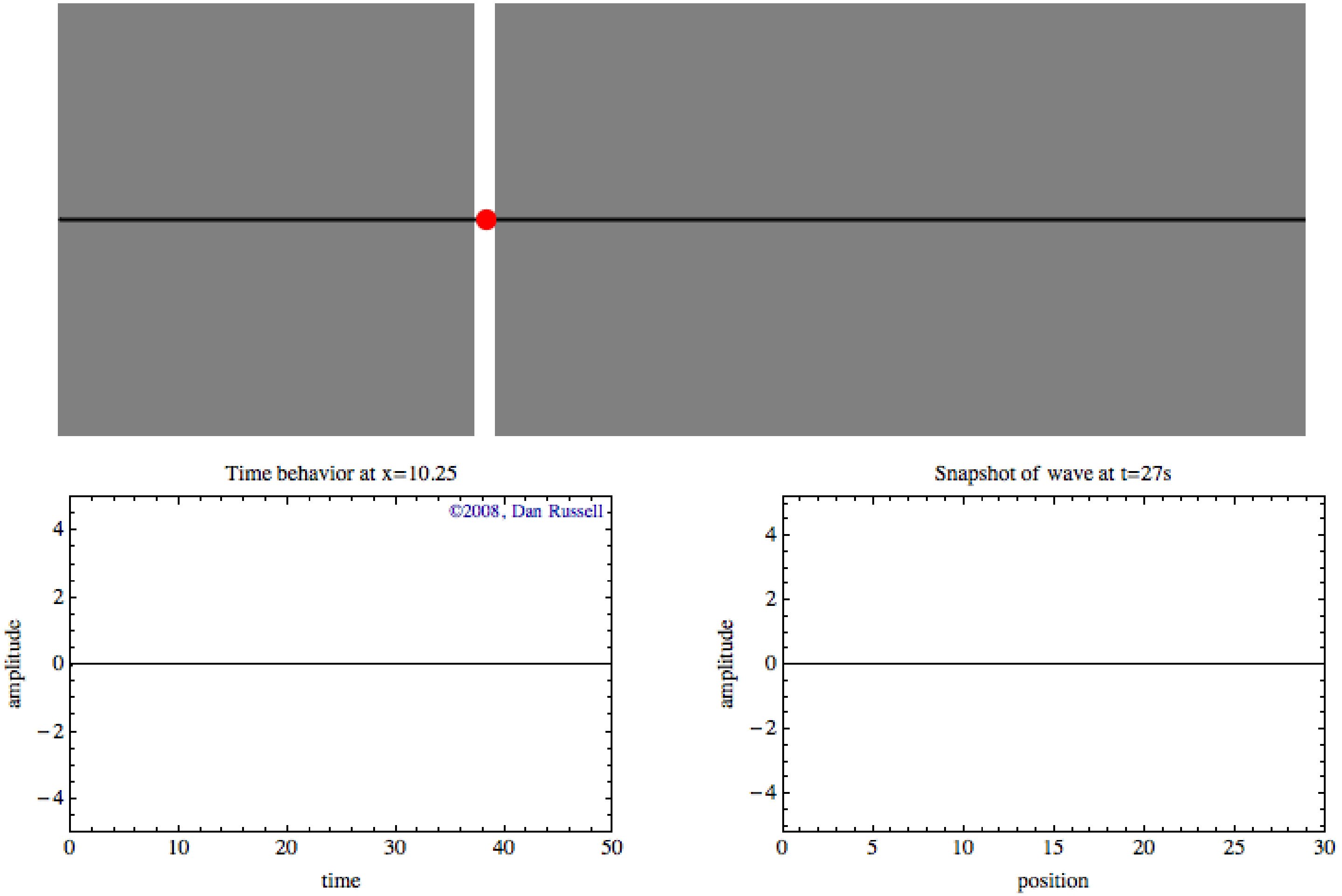
# Wave equation solution

Travelling waves – d'Alambert's solution:



# Wave equation solution

Time and position dependency: $u(x,t) = \widehat{u_+}\cos(\omega t - kx) = \widehat{u_+}e^{-i(\omega t - kx)}$

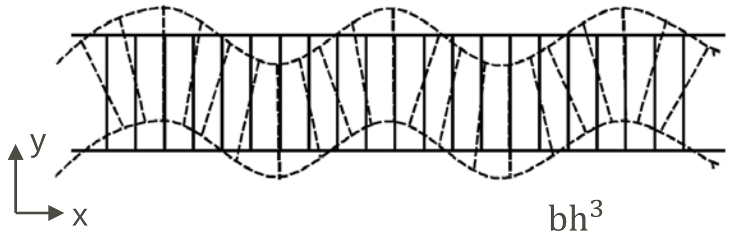


# Bending waves – reprise

- Back to bending waves to say two things:
  - One about solution to bending waves
  - One about waves in general
- Bending waves (dispersive)

$$B\frac{\partial^4 u_y}{\partial x^4} + m\frac{\partial^2 u_y}{\partial t^2} = 0 \qquad c_{B(\omega)} = \sqrt{\omega}\sqrt[4]{\frac{B}{m}}$$

Plate: E, G, ρ, υ, h




$$m = \rho h \qquad B_{beam} = E\frac{bh^3}{12}$$

$$B_{plate} = \frac{Eh^3}{12(1-\upsilon^2)}$$

- Different form of wave equation!

# Bending waves - solution

$$B\frac{\partial^4 u_y}{\partial x^4} + m\frac{\partial^2 u_y}{\partial t^2} = 0$$

- Due to the different form of equations with four-times spatial derivatives, the solution is more complex solutions including *near-field* terms

$$\zeta(x,t) = \hat{\zeta} e^{i(\omega t - k_B x)}. \qquad \zeta(x,t) = (\mathbf{A}e^{ik_B x} + \mathbf{B}e^{-ik_B x} + \mathbf{C}e^{k_B x} + \mathbf{D}e^{-k_B x})e^{i\omega t},$$

- Why near field!?
  - Because they decay rather quickly away from the *boundary* or *load point!*

# Bending waves - solution

- More complex solutions including near-field terms

$$\zeta(x,t) = \hat{\zeta} e^{i(\omega t - k_B x)}. \qquad \zeta(x,t) = (\mathbf{A}e^{ik_B x} + \mathbf{B}e^{-ik_B x} + \mathbf{C}e^{k_B x} + \mathbf{D}e^{-k_B x})e^{i\omega t},$$

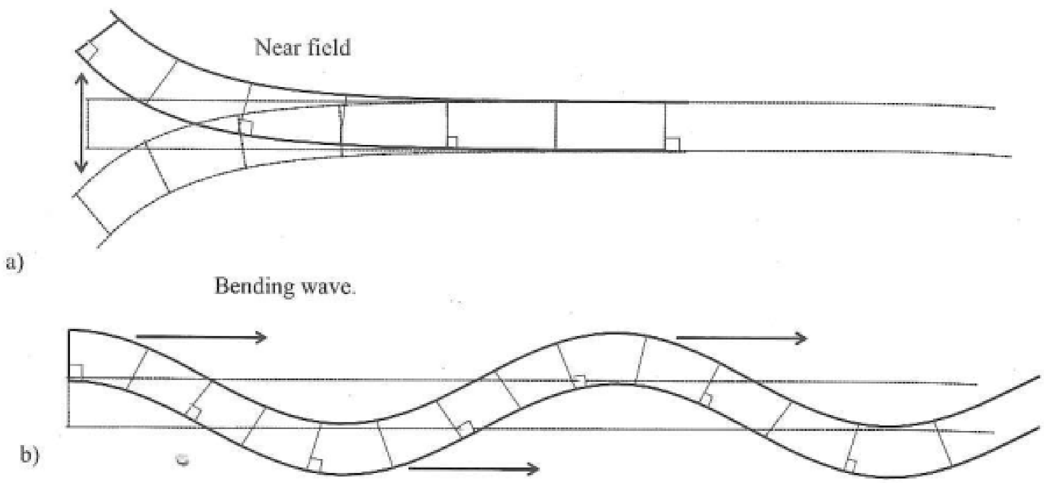



**Figure 6-23** Solutions to the bending wave equation near the end of a beam. a) The near field is characterized by an amplitude that decays exponentially with distance from the excitation point. A reasonable engineering approximation would be to ignore the near field at distances greater than 1/3 of the bending wavelength. The near field is significant near boundaries, force application points, and other discontinuities. b) The bending wave can, if the losses are small, spread over large distances.

Source: *Sound and Vibration*, Wallin, Carlsson, Åbom, Bodén, Glav

# Bending waves - solution

- More complex solutions including near-field terms

$$\zeta(x,t) = \hat{\zeta} e^{i(\omega t - k_B x)}. \qquad \zeta(x,t) = (\mathbf{A}e^{ik_B x} + \mathbf{B}e^{-ik_B x} + \mathbf{C}e^{k_B x} + \mathbf{D}e^{-k_B x})e^{i\omega t},$$

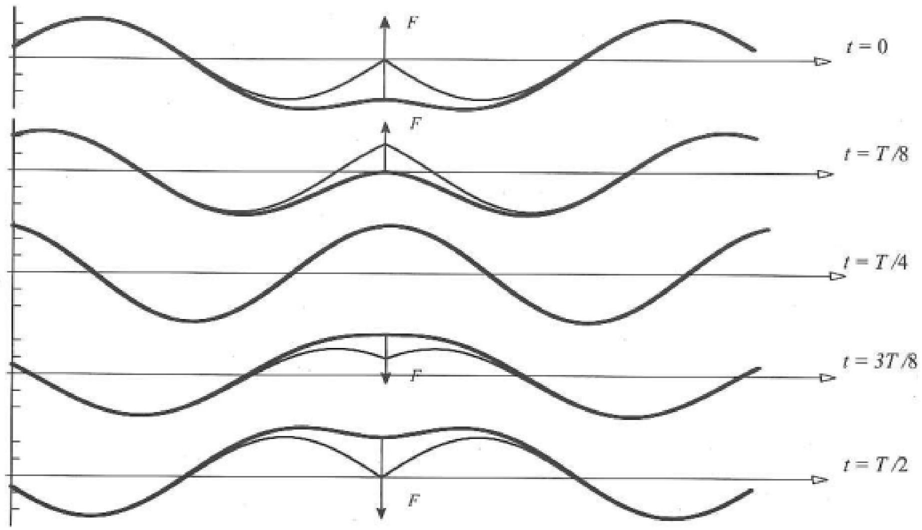



**Figure 6-24** The bending wave near field permits a continuous slope. The near field is a consequence of the beam's ability to withstand shear. The string, lacking that ability, instead exhibits a slope discontinuity at a point of force application. Thin line – no near field (string). Thick line – including near field (beam).

Source: *Sound and Vibration*, Wallin, Carlsson, Åbom, Bodén, Glav

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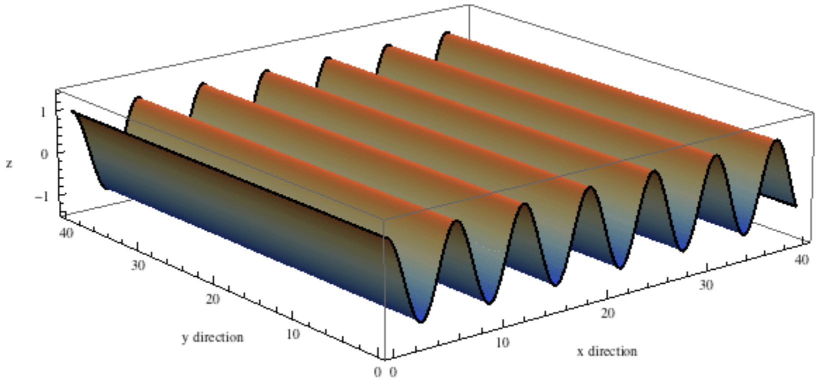
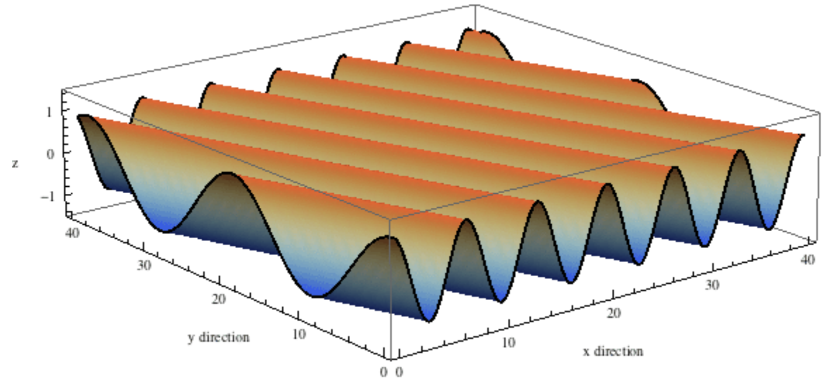
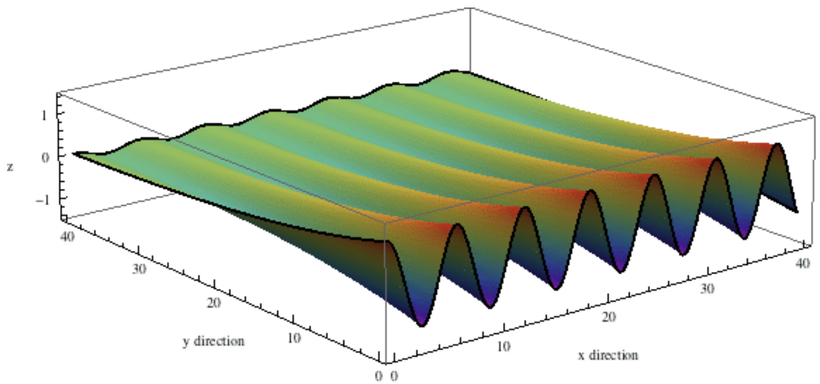
# Propagating waves VS evanescent waves

- Waves may thus propagate or not propagate!




**https://www.acs.psu.edu/drussell/Demos/EvanescentWaves/EvanescentWaves.html**

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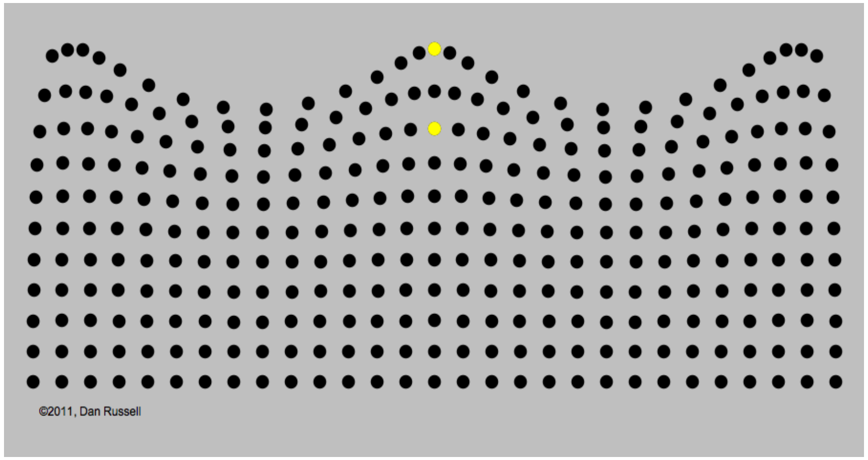
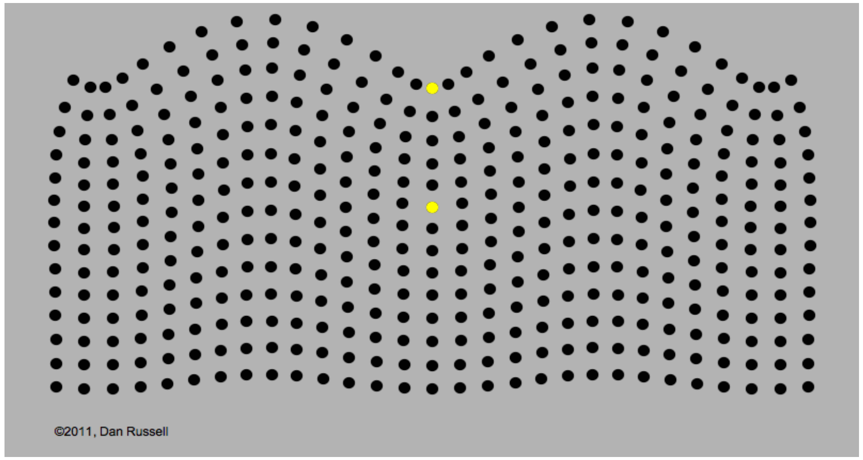
# Other types of waves…

- In reality, combinations of aforementioned waves can exist, e.g.




- <u>Surface waves</u>

**Water waves**

(long+transverse waves)

Particles in *clockwise circles*. The radius of the circles decreases increasing depth

Pure shear waves don't exist in fluids

- <u>Body waves</u>

**Rayleigh waves**

(long+transverse waves)

Particles in elliptical *paths*. Ellipses width decreases with increasing depth

Change from depth>1/5 of λ

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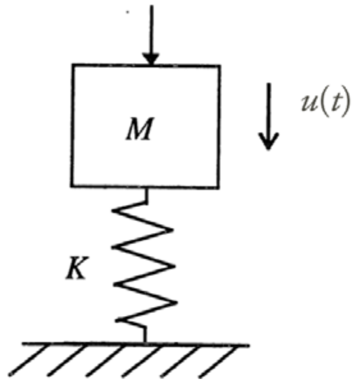
# Boundary and initial conditions

- A structure can be mathematically described by
    - Equations of motion
    - Boundary conditions (in space)
    - Initial conditions (in time – irrelevant if harmonic motion $e^{i\omega t}$ is assumed)
- Boundary-value problem – in mathematical language

# Do you remember…?

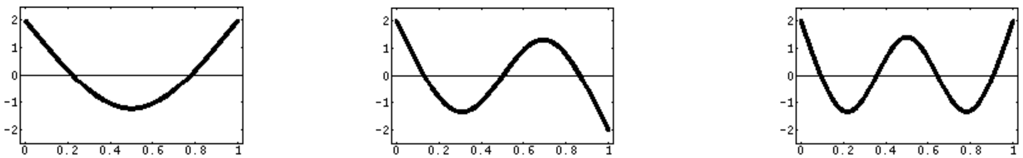



- EoM: $M\ddot{x} + Kx = 0$
- Solution: $x(t) = ae^{\mathrm{i}\omega_0 t} + be^{-\mathrm{i}\omega_0 t}$=$A\sin(\omega_0 t) + B\cos(\omega_0 t)$.
- One needs *conditions* to determine A and B.

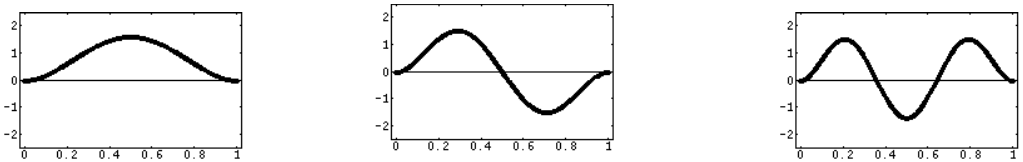
# Boundary conditions (beam in bending)

- Structures and the waves in them behave (=look) differently depending on boundary conditions – i.e. How structures are connected at their ends

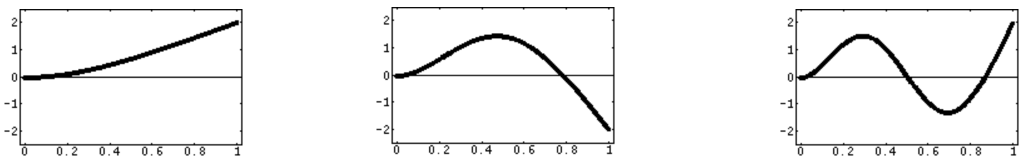
Free-free



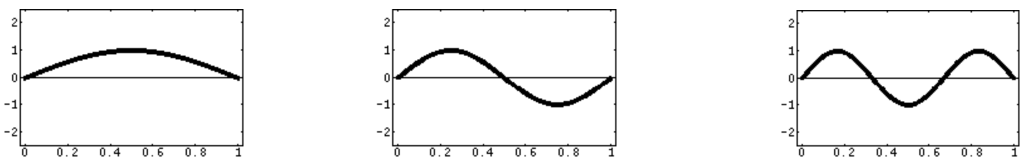

Clamped-clamped




Clamped-free




Simply supported both ends



# Summary

- Wave propagation in solid media
- Derivation of longitudinal wave equation
- Solution to wave equation
- What is a wave?
- Evanescent waves
- Boundary conditions determine shape of a wave

**REFERENCES**: Animations retrieved from Dan Russell's website

# Thank you for your attention!

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